Tempered distributions

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Intended readers: At roughly the level of a first-year graduate student of mathematics.

Specific prerequisites include parts 1 and 2 of my notes on Fourier transforms, and some introductory graduate-level analysis

Feedback: If you find this useful, or if you have comments or suggestions, or if you just want to say hello, I would very much enjoy hearing from you: cborgers@tufts.edu.

Contents

1 Purpose of these notes

This is *not* a comprehensive introduction to tempered distributions, but just a brief review of what tempered distributions are. I have added some intuition and motivation that had not previously been entirely clear to me.

All functions in these notes are assumed to be defined on $\mathbb{R}$, with values in $\mathbb{C}$. We write:

$$\mathcal{D} = C_c^\infty \quad \text{and} \quad \mathcal{S} = \text{Schwartz space.}$$

A *distribution* is a continuous linear functional on $\mathcal{D}$. The motivation is as follows. A locally integrable function $f = f(x)$ can be identified with the linear functional

$$F_f : \mathcal{D} \to \mathbb{C}, \tag{1}$$

$$\varphi \mapsto \int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx. \tag{2}$$

Knowing F_f, we can back out f. I'll very briefly review why this is so shorty. But there are also linear functionals on $\mathcal{D}$ that don't come from functions. The most famous example is

$$\varphi \mapsto \varphi(0).$$

Another interesting example is

$$\varphi \mapsto \lim_{r\to\infty} \int_{-r}^{r} \frac{\varphi(x)}{x}\, dx.$$

Distributions offer a way of viewing these examples as generalized functions.

If $\mathcal{D}$ is replaced by $\mathcal{S}$, one obtains the notion of *tempered* distributions. Then local integrability is no longer sufficient to guarantee the existence of the integrals $\int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$. One has to assume also that $f(x)$ does not grow too fast as $x \to \pm\infty$. I will discuss this, and I will discuss the reason *why* one might want to replace $\mathcal{D}$ by $\mathcal{S}$, and whether one might be able to make choices other than $\mathcal{D}$ or $\mathcal{S}$.

The notion of *continuity* of linear functionals that is used here is extremely weak, so weak that it is hard to find examples of linear functionals that are *not* continuous. Why this continuity requirement is needed at all will be discussed later in these notes; initially, it isn't needed.

2 Locally integrable functions as linear functionals on $\mathcal{D}$

Definition 1. *A measurable function $f : \mathbb{R} \to \mathbb{C}$ is called* locally integrable *if for all real a and b with $a < b$, $f \cdot 1_{[a,b]} \in L^1$. The vector space of all locally integrable functions is denoted by L^1_{loc}.*

Theorem 1. *If* $\psi \in \mathcal{D}$ *has integral 1, and if* $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)$ *for* $\varepsilon > 0$*, then*

$$\forall f \in L^1_{\text{loc}} \qquad f(x) = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} f(y)\psi_\varepsilon(x-y)\, dy \quad \text{for almost all } x.$$

This is a well-known fact of analysis. It follows from the Lebesgue differentiation theorem:

Lebesgue differentiation theorem. *Let* $f \in L^1_{\text{loc}}$*. Then*

$$f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\varepsilon} \int_{x-\varepsilon}^{x+\varepsilon} f(y)\, dy \qquad \text{for almost all } x.$$

Theorem 1 implies that the linear functional F_f (see eqs. (1), (2)) determines the function f almost everywhere. (Of course, “almost everywhere" is the best one can expect. After all, f is only *defined* almost everywhere.)

3 Locally integrable, polynomially growing functions as linear functionals on $\mathcal{S}$

Definition 2. *A function* $f \in L^1_{\text{loc}}$ *is called* polynomially growing *if there exists an* $n \in \mathbb{N}$ *and a constant* $C > 0$ *so that*

$$\limsup_{|x| \to \infty} |f(x)| \le C|x|^n.$$

The space of all such functions is denoted by $L^1_{\text{loc,poly}}$.

Theorem 2. *If* $\psi \in \mathcal{S}$ *has integral 1, and if* $\psi_\varepsilon(x) = \frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)$ *for* $\varepsilon > 0$*, then*

$$\forall f \in L^1_{\text{loc,poly}} \qquad f(x) = \lim_{\varepsilon \to 0^+} \int_{-\infty}^{\infty} f(y)\psi_\varepsilon(x-y)\, dy \quad \text{for almost all } x.$$

This, too, follows from the Lebesgue differentiation theorem: It implies that the linear functional

$$\begin{aligned} F_f : \mathcal{S} &\to \mathbb{C}. \\ \varphi &\mapsto \int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx \end{aligned}$$

determines the function f almost everywhere. The assumption of (at most) polynomial growth is needed to ensure that the integrals $\int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$, $\varphi \in \mathcal{S}$, exist.

4 How about domains other than $\mathcal{D}$ and $\mathcal{S}$?

One could consider other spaces on which to study linear functionals. I will think a bit about possible subspaces $\mathcal{V} \subseteq \mathcal{S}$ other than $\mathcal{D}$ and $\mathcal{S}$. The functions in $\mathcal{V}$ are called *test functions* in this context.

Notation:

$\mathcal{V}^*$ = *space of linear functionals* $\mathcal{V} \to \mathbb{C}$ *(without any continuity condition)*

In light of Theorem 2, one might consider taking $\mathcal{V}$ to be the much smaller space

$$\mathcal{G} = \text{span}\left\{e^{-(x-\mu)^2/(2\sigma^2)} \;:\; \mu \in \mathbb{R}, \quad \sigma > 0\right\}$$

for instance. The space $L^1_{\text{loc,poly}}$ is embedded in $\mathcal{G}^*$, in the sense that the functional

$$\varphi \mapsto \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx$$

determines the function f almost everywhere if $f \in L^1_{\text{loc,poly}}$. Actually, f could even be something like e^x here, which grows faster than polynomially as $x \to \infty$. Reducing the domain, $\mathcal{V}$, of the linear functionals can broaden the space of functions that can be viewed as embedded in $\mathcal{V}^*$.

5 Derivatives of linear functionals

Definition 3. *Let* $\mathcal{V} \subseteq \mathcal{S}$ *be a subspace that is invariant under differentiation:*

$$\forall \varphi \in \mathcal{V} \qquad \varphi' \in \mathcal{V}.$$

Let $F \in \mathcal{V}^*$. *The* derivative of F *is the linear functional* $F' \in \mathcal{V}^*$ *defined by*

$$\langle F', \varphi \rangle = -\langle F, \varphi' \rangle .$$

This is motivated by integration by parts.

Evidently $\mathcal{D}$ and $\mathcal{S}$ are invariant under differentiation, whereas $\mathcal{G}$ is not. Derivatives of linear functionals on $\mathcal{G}$ are not well-defined. However, the following larger spaces are invariant under differentiation:

$$\mathcal{H} = \text{span}\left\{x^n e^{-(x-\mu)^2/(2\sigma^2)} \;:\; n \in \mathbb{N} \cup \{0\}, \quad \mu \in \mathbb{R}, \quad \sigma > 0\right\},$$

and

$$\mathcal{I} = \text{span}\left\{x^n e^{-(x-\mu)^2/(2\sigma^2)} \;:\; n \in \mathbb{N} \cup \{0\}, \quad \mu \in \mathbb{C}, \quad \sigma > 0\right\}.$$

Linear functionals in $\mathcal{D}^*$, $\mathcal{S}^*$, $\mathcal{H}^*$, $\mathcal{I}^*$ can be differentiated infinitely often.

6 Fourier transforms of linear functionals

Definition 4. *Let $\mathcal{V} \subseteq \mathcal{S}$ be a subspace that is invariant under Fourier transform:*

$$\forall \varphi \in \mathcal{V} \quad \hat{\varphi} \in \mathcal{V}.$$

Let $F \in \mathcal{V}^*$. The Fourier transform of F *is the linear functional $\hat{F} \in \mathcal{V}^*$ defined by*

$$\left\langle \hat{F}, \varphi \right\rangle = \langle F, \hat{\varphi} \rangle .$$

This is motivated by the symmetry property of the Fourier transform on L^1.

Theorem 3. *$\mathcal{D} = C_c^\infty$ is not invariant under Fourier transform. Much more strongly:*

$$\varphi \in \mathcal{D} - \{0\} \;\Rightarrow\; \hat{\varphi} \notin \mathcal{D}.$$

This is the reason why $\mathcal{D}$ is not a good domain for distributions when one wants to work with Fourier transforms.

Proof. Let $\varphi \in \mathcal{D}$. Then

$$\hat{\varphi}(t) = \int_{-\infty}^{\infty} \varphi(x) \left(\sum_{k=0}^{\infty} \frac{(ixt)^k}{k!} \right) dx = \int_{-\infty}^{\infty} \sum_{k=0}^{\infty} \varphi(x) \frac{(ix)^k}{k!} t^k \, dx.$$

The integral and sum can be swapped; this follows for instance from Fubini's theorem. So we obtain

$$\hat{\varphi}(t) = \sum_{k=0}^{\infty} \left(\int_{-\infty}^{\infty} \varphi(x) \frac{(ix)^k}{k!} \, dx \right) t^k.$$

This shows that $\hat{\varphi}$ is analytic. It therefore cannot vanish on an interval unless it vanishes everywhere. □

It is advantageous to keep the domain of the linear functionals small. For instance, I noted earlier that the space of functions that can be thought of as elements of $\mathcal{V}^*$ may be larger for smaller $\mathcal{V}$. To obtain a small space that is invariant under Fourier transform, starting with $\mathcal{D}$, we might add $\mathcal{F}[\mathcal{D}]$. We define

$$\mathcal{R} = \mathcal{D} + \mathcal{F}[\mathcal{D}].$$

Theorem 4.

space	invariant under differentiation?	invariant under Fourier transform?
$\mathcal{D}$	✓	X
$\mathcal{R}$	✓	✓
$\mathcal{G}$	X	X
$\mathcal{H}$	✓	X
$\mathcal{I}$	✓	✓
$\mathcal{S}$	✓	✓

Proof. (1) It is clear that $\mathcal{D}$ is invariant under differentiation. We know from Theorem 3 that is is not invariant under Fourier transform.

(2) We will next show that $\mathcal{D}+\mathcal{F}[\mathcal{D}]$ is invariant under differentiation. Since $\mathcal{D}$ is invariant under differentiation, it suffices to show that the derivative of a function in $\mathcal{F}[\mathcal{D}]$ belongs to $\mathcal{F}[\mathcal{D}]$. So let $\psi \in \mathcal{D}$ and $\varphi = \mathcal{F}[\psi] = \hat{\psi}$. Then

$$\varphi'(t) = \int_{-\infty}^{\infty} \psi(x)\; ix\, e^{ixt}\; dx,$$

which is the Fourier transform of $ix\psi(x) \in \mathcal{D}$. To show that $\mathcal{D}+\mathcal{F}[\mathcal{D}]$ is invariant under Fourier transform, note that the Fourier transform of a function in $\mathcal{D}$ is in $\mathcal{F}[\mathcal{D}]$, and the Fourier transform of $\mathcal{F}[\mathcal{D}]$ is, by the inversion theorem, in $\mathcal{D}$.

(3) Clearly $\mathcal{G}$ is not invariant under differentiation. It is not invariant under Fourier transform either. For instance,

$$\mathcal{F}\left[e^{-(x-1)^2}\right] = \int_{-\infty}^{\infty} e^{-(x-1)^2} e^{ixt}\; dx = e^{it}\int_{-\infty}^{\infty} e^{-(x-1)^2} e^{i(x-1)t}\; dx =$$

$$e^{it}\int_{-\infty}^{\infty} e^{-x^2} e^{ixt}\; dx = \int_{-\infty}^{\infty} e^{-(x-it/2)^2}\; dx\; e^{-t^2/4}\; e^{it} = \sqrt{\pi}\, e^{-t^2/4}\; e^{it} \notin \mathcal{G}.$$

(Justification of the last equation involves contour integration.)

(4) $\mathcal{H}$ is obviously invariant under differentiation, but it isn't invariant under Fourier transform either:

$$\mathcal{F}\left[e^{-(x-1)^2}\right] = \sqrt{\pi}\, e^{-t^2/4}\; e^{it} \notin \mathcal{H}.$$

(5) $\mathcal{I}$ is obviously invariant under differentiation as well. To prove that it is also invariant under Fourier transform, we must show that the Fourier transform of

$$\varphi(x) = x^n e^{-(x-\mu)^2/(2\sigma^2)},$$

where $n \in \mathbb{N}\cup\{0\}$, $\mu \in \mathbb{C}$, and $\sigma > 0$, belongs to $\mathcal{I}$. We have

$$\hat{\varphi}(t) = \int_{-\infty}^{\infty} x^n e^{-(x-\mu)^2/(2\sigma^2)}\; e^{ixt}\; dx = (-i)^n \frac{d^n}{dt^n}\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\; e^{ixt}\; dx.$$

Since $\mathcal{I}$ is invariant under differentiation, it suffices now to prove that

$$\int_{-\infty}^{\infty} e^{-(x-\mu)^2/(2\sigma^2)}\ e^{ixt}\ dx \in \mathcal{I}.$$

We write this as

$$\int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma^2}-\frac{\mu x}{\sigma^2}-ixt+\frac{\mu^2}{2\sigma^2}\right)}\ dx \in \mathcal{I}.$$

Complete the square:

$$\frac{x^2}{2\sigma^2}-\frac{\mu x}{\sigma^2}-ixt+\frac{\mu^2}{2\sigma^2}=\frac{1}{2\sigma^2}\left(x^2-2\mu x-2\sigma^2 ixt+\mu^2\right)=$$

$$\frac{1}{2\sigma^2}\left[\left(x-(\mu+\sigma^2 it)\right)^2+\mu^2-(\mu+\sigma^2 it)^2\right]=\frac{1}{2\sigma^2}\left[\left(x-(\mu+\sigma^2 it)\right)^2-2\mu\sigma^2 it+\sigma^4 t^2\right]=$$

$$\frac{(x-(\mu+\sigma^2 it))^2}{2\sigma^2}-\mu it+\frac{\sigma^2 t^2}{2}.$$

So

$$\int_{-\infty}^{\infty} e^{-\left(\frac{x^2}{2\sigma^2}-\frac{x\mu}{\sigma^2}-ixt+\frac{\mu^2}{2\sigma^2}\right)}\ dx = e^{-\frac{\sigma^2 t^2}{2}+\mu it}\int_{-\infty}^{\infty} e^{-\frac{\left(x-(\mu+\sigma^2 it)\right)^2}{2\sigma^2}}\ dx. \tag{3}$$

Using contour integration, and a change of variables, the integral is seen to equal $\sqrt{2\pi\sigma^2}$. Altogether, (3) becomes

$$\sqrt{2\pi\sigma^2}\ e^{-\frac{\sigma^2 t^2}{2}+\mu it} \in \mathcal{I}.$$

(6) $\mathcal{S}$ is invariant under differentiation and Fourier transform; see the second part of my notes on the Fourier transform. □

So differentiation and Fourier transforms can be defined on $\mathcal{R}^*$ or on $\mathcal{I}^*$. The space $L^1_{\text{loc,poly}}$ can be thought of as a subspace of both $\mathcal{R}^*$ and $\mathcal{I}^*$. More functions than those in $L^1_{\text{loc,poly}}$ can be thought of as elements of $\mathcal{I}^*$ — for instance, the function e^x. Interestingly, $\mathcal{R}$ and $\mathcal{I}$ have no non-zero elements in common:

Theorem 5. $\mathcal{R}\cap\mathcal{I}=\{0\}$.

Proof. Suppose $\varphi\in\mathcal{I}$ can be written in the form

$$\varphi=\psi+\hat{\eta}$$

where $\psi\in\mathcal{D}$ and $\eta\in\mathcal{D}$. All elements of $\mathcal{I}$ are analytic functions. Also, $\hat{\eta}$ is analytic; see the proof of Theorem 3. Therefore

$$\psi=\varphi-\hat{\eta}$$

is analytic. Since ψ is zero outside a compact interval, $\psi = 0$. So

$$\varphi = \hat{\eta}.$$

By the inversion formula,

$$\hat{\varphi} = \hat{\hat{\eta}} = 2\pi\eta(-x).$$

Since $\mathcal{I}$ is invariant under Fourier transform, this implies $2\pi\eta(-x) \in \mathcal{I}$, so $\eta \in \mathcal{I}$. Therefore η is both analytic, and has compact support, therefore $\eta = 0$, and hence $\varphi = 0$. □

7 Convolving linear functionals with test functions

Definition 5. *Let $\mathcal{V}$ be a subspace of $\mathcal{S}$. Assume*

$$\forall \varphi \in \mathcal{V} \qquad \forall x \in \mathbb{R} \quad \varphi(x - \cdot) \in \mathcal{V}. \tag{4}$$

Let $F \in \mathcal{V}^*$ and $\varphi \in \mathcal{V}$. The convolution $F * \varphi$ *is the function*

$$(F * \varphi)(x) = \langle F(y), \varphi(x - y) \rangle. \tag{5}$$

I wrote "$F(y)$" in (5) *to indicate that F is acting on $\varphi(x-y)$ thought of as a function of y, with x fixed.*

The spaces $\mathcal{D}$, $\mathcal{R}$, $\mathcal{I}$, and $\mathcal{S}$ all satisfy the condition (4).

8 The first place where continuity matters: smoothness of convolutions

Let's try to prove differentiability of $F * \varphi$. We have

$$\frac{\langle F(y), \varphi(x+h-y)\rangle - \langle F(y), \varphi(x-y)\rangle}{h} = \left\langle F(y), \frac{\varphi(x+h-y) - \varphi(x-y)}{h} \right\rangle.$$

The functions

$$\frac{\varphi(x+h-y) - \varphi(x-y)}{h}, \quad y \in \mathbb{R},$$

with x fixed, converge, as $h \to 0$, to $\varphi'(x-y)$. This is obviously valid pointwise, but it is valid also in a much stronger sense, to be discussed momentarily. To conclude that $F * \varphi$ is differentiable, we'd like to say now that

$$\lim_{h\to 0} \left\langle F(y), \frac{\varphi(x+h-y) - \varphi(x-y)}{h} \right\rangle = \left\langle F(y), \lim_{h\to 0} \frac{\varphi(x+h-y) - \varphi(x-y)}{h} \right\rangle = \langle F(y), \varphi'(x-y) \rangle.$$

For this to be valid, we need a continuity condition on F.

Definition 6. *A sequence* $\{\varphi_k\}_{k\in\mathbb{N}}$ *of Schwartz functions* converges to 0 in $\mathcal{S}$ *if*

$$\forall \alpha \in \mathbb{N}\cup\{0\} \quad \forall \beta \in \mathbb{N}\cup\{0\} \qquad \lim_{k\to\infty} \left\| x^{\alpha} \frac{d^{\beta}\varphi_k}{dx^{\beta}} \right\|_{\infty} = 0 \tag{6}$$

A sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ *of functions in* $\mathcal{S}$ converges to $\lambda \in \mathcal{S}$ *if the sequence* $\{\varphi_k - \lambda\}_{k\in\mathbb{N}}$ *converges to zero in* $\mathcal{S}$*.*

Lemma 1. *As* $h \to 0$*, the functions*

$$\frac{\varphi(x+h-y)-\varphi(x-y)}{h}$$

(understood as functions of y*, with* x *fixed) converge to* $\varphi'(x-y)$ *in* $\mathcal{S}$*.*

Proof. This is straightforward. As an example, I'll spell out the proof that (for a fixed x)

$$\lim_{h\to 0} \left\| y\,\frac{d}{dy}\left(\frac{\varphi(x+h-y)-\varphi(x-y)}{h} - \varphi'(x-y)\right)\right\|_{\infty} = 0.$$

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Let $|h| \le 1$. Then

$$\begin{aligned}
&\left| y\,\frac{d}{dy}\left(\frac{\varphi(x+h-y)-\varphi(x-y)}{h} - \varphi'(x-y)\right)\right| \\
= \quad & |y| \left| \frac{\varphi'(x+h-y)-\varphi'(x-y)}{h} - \varphi''(x-y)\right| = \\
= \quad & |y|\ |\varphi''(x+ch-y) - \varphi''(x-y)| \qquad \text{(for some } c \in (0,1)\text{)} \\
= \quad & |y\ \varphi'''(x+cdh-y)|\ |h| \qquad \text{(for some } d \in (0,1)\text{)} \\
\le \quad & (|(x+cdh-y)\ \varphi'''(x+cdh-y)| + |(x+cdh)\ \varphi'''(x+cdh-y)|)\ |h| \\
\le \quad & (\|s\ \varphi'''(s)\|_{\infty} + (|x|+1)\,\|\varphi'''(s)\|_{\infty})\ |h|.
\end{aligned}$$

(In the last step, I used that $|h| \le 1$. I used the letter s for the independent variable because x and y are taken.) So

$$\left\| y\,\frac{d}{dy}\left(\frac{\varphi(x+h-y)-\varphi(x-y)}{h} - \varphi'(x-y)\right)\right\|_{\infty} \le$$

$$(\|s\ \varphi'''(s)\|_{\infty} + (|x|+1)\,\|\varphi'''(s)\|_{\infty})\ |h|.$$

This tends to zero as $h \to 0$. □

Definition 7. *Let $\mathcal{V} \subseteq \mathcal{S}$ be a subspace. A linear functional $F \in \mathcal{V}^*$ is called a* distribution *on $\mathcal{V}$ if it is continuous in the sense that $\varphi_k \to 0$ in $\mathcal{S}$ (in the sense of Definition 6) implies $F(\varphi_k) \to 0$. The space of all continuous linear functionals on $\mathcal{V}$ is denoted by $\mathcal{V}'$.*

The distributions on $\mathcal{S}$ are called tempered distributions.

Definition 6 describes an extremely strong notion of convergence. Since the notion of convergence in $\mathcal{S}$ is so strong, the notion of continuity is extremely weak. It is, however, sufficient for our purposes. We conclude from the preceding discussion:

Theorem 6. *Let $\mathcal{V} \subseteq \mathcal{S}$ be a subspace that is invariant with respect to differentiation. Assume*

$$\forall \varphi \in \mathcal{V} \qquad \forall x \in \mathbb{R} \qquad \varphi(x - \cdot) \in \mathcal{V}.$$

Let $F \in \mathcal{V}'$ and $\varphi \in \mathcal{V}$. Then

$$\frac{d}{dx}(F * \varphi)(x) = (F * \varphi')(x),$$

and consequently more generally

$$\frac{d^k}{dx^k}(F * \varphi)(x) = F * \frac{d^k\varphi}{dx^k}(x)$$

for all $k \geq 0$. In particular, $F * \varphi \in C^\infty$.

9 A discontinuous linear functional on the Schwartz space

Here is a fairly artificial and non-explicit construction of a linear functional on $\mathcal{S}$ that is not continuous. Let $(b_i)_{i \in I}$ be a Hamel basis of $\mathcal{S}$. Let $\varphi_1, \varphi_2, \ldots \in \mathcal{S}$ with $\varphi_k = b_{i_k}$, $i_k \in I$, and the i_k are different from each other. By scaling the φ_k, we can assume without loss of generality that

$$\forall \alpha \in \{0, 1, \ldots, k\} \qquad \forall \beta \in \{0, 1, \ldots, k\} \qquad \left\| x^\alpha \frac{d^\beta \varphi_k}{dx^\beta} \right\|_\infty \leq \frac{1}{k}.$$

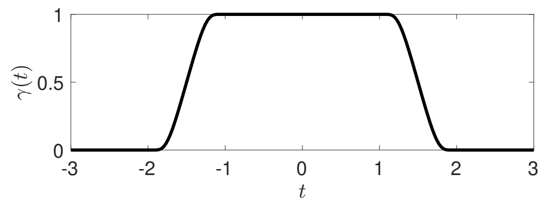
Then clearly $\varphi_k \to 0$ in $\mathcal{S}$. Now define a linear functional F on $\mathcal{S}$ by $F(\varphi_k) = 1$, and $F(b_i) = 0$ if b_i if none of the φ_k. Clearly, F is not continuous.

10 Tempered distributions are limits of functions in $\mathbf{C_c^\infty}$

Theorem 7. *For any $F \in \mathcal{S}'$, there exists a sequences $\eta_1, \eta_2, \ldots$ in C_c^∞ with*

$$\forall \varphi \in \mathcal{S} \quad \langle F, \varphi \rangle = \lim_{k\to\infty} \int_{-\infty}^{\infty} \eta_k(x)\varphi(x)\, dx. \tag{7}$$

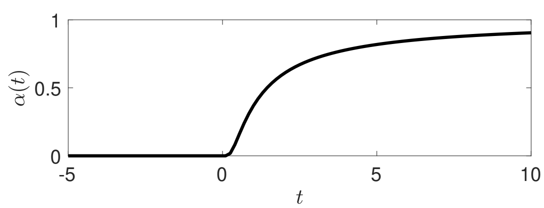
Sketch of proof. Make F smooth by convolving with $\frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)$ for a small $\varepsilon > 0$, where $\psi \in \mathcal{S}$ with $\int_{-\infty}^{\infty} \psi(x)\, dx = 1$. Then make the support compact by multiplying by $\gamma(x/R)$ for a large R, where $\gamma(x) = 1$ for $|x| \le 1$ $\gamma(x) = 0$ for $|x| \ge 2$, and γ is monotonic in $[-2, -1]$ and in $[1, 2]$.



□

Here is a way of constructing γ. We start with

$$\alpha(t) = \begin{cases} 0 & \text{if } t \ge 0, \\ e^{-1/t} & \text{if } t > 0. \end{cases}$$



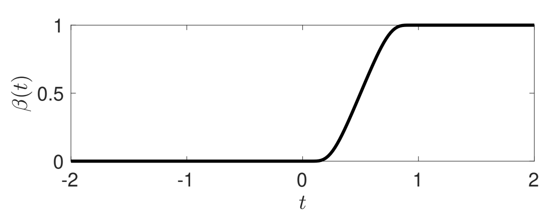
then define

$$\beta(t) = \frac{\alpha(t)}{\alpha(t) + \alpha(1-t)}.$$

Then $\beta(t) = 0$ for $t \le 0$, $\beta(t) = 1$ for $t \ge 1$, and β is increasing, since for $t > 0$ we can write

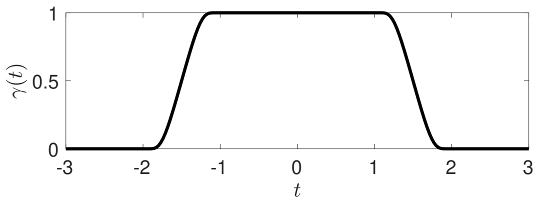
$$\beta(t) = \frac{1}{1 + \alpha(1-t)/\alpha(t)}.$$

($\alpha(1-t)$ decreases, $\alpha(t)$ increases, so $\alpha(1-t)/\alpha(t)$ decreases, and β increases.)



Then define

$$\gamma(t) = \beta(t+2)\beta(-t+2).$$



It is clear that the convergence in (7) cannot be uniform in φ. For instance, think of $F = 1$. That's a tempered distribution, and convolving with $\frac{1}{\varepsilon}\psi\left(\frac{x}{\varepsilon}\right)$ doesn't change it at all. So a function in $\mathcal{D}$ that approximates F, in this case, is simply the function $\gamma(x/R)$ for large R. That doesn't approximate $\langle F, \varphi\rangle$ for *all* φ, but it does approximate it well if $\varphi(x)$ is very close to zero for $|x| \geq R$.

11 The Fourier transform of x²

The function x^2 is a tempered distribution, which we'll denote by F. Being a tempered distribution, it has a Fourier transform. And the strangest thing is, that Fourier transform can approximately be thought of as a smooth, compactly supported function.

Why is that, in this example?

$$\langle \hat{F}(t), \varphi(t)\rangle = \langle F(x), \hat{\varphi}(x)\rangle = \int_{-\infty}^{\infty} x^2 \, \hat{\varphi}(x) \, dx.$$

Now

$$x^2\hat{\varphi}(x) = x^2 \int_{-\infty}^{\infty} \varphi(t) e^{itx} \, dt = -\int_{-\infty}^{\infty} \varphi(t) \frac{d^2}{dt^2} e^{itx} \, dt = -\int_{-\infty}^{\infty} \varphi''(t) e^{itx} \, dt.$$

So we can also write

$$\langle \hat{F}(t), \varphi(t)\rangle = -\int_{-\infty}^{\infty} \widehat{\varphi''}(x) \, dx.$$

This, by the inversion formula, is

$$-2\pi\varphi''(0).$$

Here is the result, in summary. The tempered distribution

$$\varphi \mapsto \int_{-\infty}^{\infty} x^2 \varphi(x) \, dx$$

has the Fourier transform

$$\varphi \mapsto -2\pi\varphi''(0).$$

To approximate this by a smooth function, we convolve it with

$$\psi_\varepsilon(x) = \frac{e^{-(x/\varepsilon)^2}}{\varepsilon\sqrt{\pi}}$$

for a small $\varepsilon > 0$. We obtain

$$\langle \hat{F}(s), \psi_\varepsilon(t-s) \rangle = -2\pi \psi''_\varepsilon(t) = \frac{4\sqrt{\pi}}{\varepsilon^3} \left(1 - \frac{2}{\varepsilon^2}\, t^2\right)\, e^{-(t/\varepsilon)^2}.$$

In summary:

The Fourier transform of the function x^2, mollified by convolving with

$$\psi_\varepsilon(x) = \frac{e^{-(x/\varepsilon)^2}}{\varepsilon\sqrt{\pi}},$$

$\varepsilon > 0$*, equals*

$$\frac{4\sqrt{\pi}}{\varepsilon^3} \left(1 - \frac{2}{\varepsilon^2}\, t^2\right) e^{-t^2/\varepsilon^2}.$$