Fourier transforms, part 2: Fourier transforms of Schwartz functions

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Intended readers: At roughly the level of a first-year graduate student of mathematics.

Specific prerequisites include part 1 of my notes on Fourier transforms

Feedback: If you find this useful, or if you have comments or suggestions, or if you just want to say hello, I would very much enjoy hearing from you: cborgers@tufts.edu.

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1 Overview

At first sight, it seems most natural to define the Fourier transform on L^1. But then the range of the Fourier transform becomes the Wiener algebra A, a subspace of C_0 — not all functions in L^1 belong to A, and not functions in A belong to L^1.

It turns out to be a good idea to narrow our focus very slightly, to the space $\mathcal{S}$ of *Schwartz functions.* These are functions that are infinitely often differentiable, with the property that the function itself and all of its derivatives decay faster than any polynomial at infinity. (See Section 2 for details.) In fact, $\mathcal{S}$ is dense in L^1, as well as dense in C_0, and contained in A. So L^1-functions are indistinguishable to the eye from Schwartz functions, and so are functions in C_0. It turns out that

$$\mathcal{F}: \mathcal{S} \to \mathcal{S}$$

is a bijection.

2 Schwartz space

Schwartz functions are named after Laurent Schwartz, who was French and lived mostly in the 20th century (1915-2002). He is not to be confused with Hermann Schwarz, who was

German, lived mostly in the 19th century (1843–1921), has the Cauchy-Schwarz inequality named after him, and has no t in front of the z in his name.

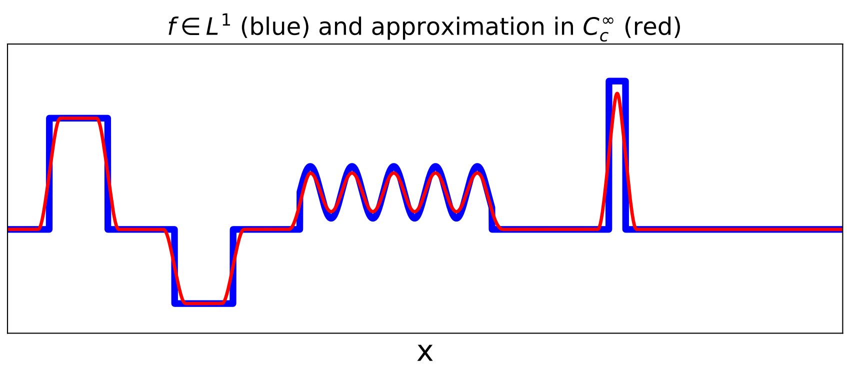
Definition 1. *A function $f \in C^\infty$ is called a* Schwartz function *if*

$$\forall \alpha \in \mathbb{N} \cup \{0\} \quad \forall \beta \in \mathbb{N} \cup \{0\} \qquad \lim_{|x| \to \infty} \left(x^\alpha \frac{d^\beta f}{dx^\beta} \right) = 0.$$

The vector space of all Schwartz functions is denoted by $\mathcal{S}$ and is called the Schwartz space*.*

$\mathcal{S}$ is an *algebra:* Products of Schwartz functions are Schwartz functions. Also, obviously derivatives of Schwartz functions are Schwartz functions. Examples of Schwartz functions are the functions in C_c^∞, and the Gaussians.

Schwartz functions are useful because, to the naked eye, any L^p function with $1 \le p < \infty$, and any function in C_0 looks like a Schwartz function, and in fact like a function in C_c^∞.



Theorem 1. *C_c^∞ is dense in all L^p-spaces with $1 \le p < \infty$, as well as in C_0.*

Sketch of proof. First approximate f by $f \cdot 1_{[-R,R]}$ with $R > 0$ large enough. Then convolve with a function of the form

$$\frac{1}{\delta} \varphi\left(\frac{x}{\delta}\right)$$

where $\varphi \in C_c^\infty$, $\int_{-\infty}^{\infty} \varphi(x) dx = 1$, and $\delta > 0$ is small enough. □

3 The Fourier transform as an automorphism of the Schwartz space

Theorem 2. *The Fourier transform is a bijection*

$$\mathcal{F}: \mathcal{S} \to \mathcal{S}.$$

Proof. We have to prove first that Fourier transforms of Schwartz functions are Schwartz functions. So let $f \in \mathcal{S}$. Let $\alpha \in \mathbb{N} \cup \{0\}$ and $\beta \in \mathbb{N} \cup \{0\}$. We consider

$$t^{\alpha} \frac{\partial^{\beta} \hat{f}}{\partial t^{\beta}} = t^{\alpha} \int_{-\infty}^{\infty} f(x)(ix)^{\beta} e^{itx}\, dx = \frac{1}{i^{\alpha}} \int_{-\infty}^{\infty} f(x)(ix)^{\beta} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(e^{itx}\right)\, dx =$$

$$\int_{-\infty}^{\infty} i^{\alpha} \frac{\partial^{\alpha}}{\partial x^{\alpha}} \left(f(x)(ix)^{\beta}\right)\, e^{itx}\, dx \tag{1}$$

(**??**) is the Fourier transform of a Schartz function, and therefore of an L^1-function. As $|t| \to \infty$, it tends to zero by the Riemann-Lebesgue lemma.

Next we have to prove that every Schwartz function is the Fourier transform of a Schwartz function. Let $g \in \mathcal{S}$. Then

$$\forall t \in \mathbb{R} \qquad g(t) = \frac{1}{2\pi} \hat{\hat{g}}(-t)$$

by the inversion theorem. This equals

$$\mathcal{F}\left[\frac{\hat{g}(-x)}{2\pi}\right](t).$$

So g is the Fourier transform of the Schwartz function $\frac{\hat{g}(-x)}{2\pi}$.

We know that $\mathcal{F}$ is injective, and the proof is therefore complete.

□

4 Why it isn't enough to consider rapid decay of the function alone

Proposition 1. *Let*

$$f(x) = \cos\left(e^{x^2}\right) e^{-x^2}.$$

Then

$$t\hat{f}(t) \notin L^1.$$

The conclusion is that a function that decays faster than $1/|x|^{\alpha}$ for any $\alpha \in \mathbb{N} \cup \{0\}$ need not have a Fourier transform with the same property.

Proof. Suppose that $t\hat{f} \in L^1$. Then also $\hat{f} \in L^1$, and

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}\,dt$$

for all x. Differentiating under the integral sign, we obtain

$$f'(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} (-it)\hat{f}(t)e^{-itx}\,dt.$$

So f' is the Fourier transform of an L^1-function, and therefore

$$\lim_{|x|\to\infty} f'(x) = 0.$$

But

$$f'(x) = -2x\cos\left(e^{x^2}\right)e^{-x^2} - 2x\sin\left(e^{x^2}\right)$$

does not converge to zero as $|x| \to \infty$. □