# Fourier transforms, part 3:
# Fourier transforms of square integrable functions

Christoph Börgers
Department of Mathematics, Tufts University
January 2026

**Intended readers:** At roughly the level of a first-year graduate student of mathematics.

**Specific prerequisites include** parts 1 and 2 of my notes on Fourier transforms

**Feedback:** If you find this useful, or if you have comments or suggestions, or if you just want to say hello, I would very much enjoy hearing from you: cborgers@tufts.edu.

## Contents



## 1 A simple fact about Fourier transforms and complex conjugates

The following fact is completely straightforward from the definition of $\mathcal{F}$, but I find it worth recording since it will be used several times.

**Lemma 1.** *Let $f \in L^1$. Then*
$$\mathcal{F}[\overline{f}](t) = \overline{\mathcal{F}[f]}(-t).$$

The Fourier transform of the complex conjugate is the complex conjugate of the Fourier transform, but evaluated at $-t$, not at $t$.

## 2 The Fourier transform on $\mathcal{S}$ preserves $L^2$-products up to $2\pi$

**Theorem 1.** *Let $f \in \mathcal{S}$ and $g \in \mathcal{S}$. Then*

$$\int_{-\infty}^{\infty} \hat{f}(t)\ \overline{\hat{g}(t)}\ dt = 2\pi \int_{-\infty}^{\infty} f(x)\ \overline{g(x)}\ dx.$$

*Proof.*

$$\int_{-\infty}^{\infty} \hat{f}(t)\ \overline{\hat{g}(t)}\ dt = \int_{-\infty}^{\infty} \mathcal{F}[f](t)\mathcal{F}[\overline{g}](-t)\ dt.$$

By symmetry, this is

$$\int_{-\infty}^{\infty} f(x)\ \mathcal{F}[\mathcal{F}[\overline{g}](-t)](x)\ dx.$$

By Lemma 1, this is

$$\int_{-\infty}^{\infty} f(x)\ \mathcal{F}\left[\overline{\mathcal{F}[g]}\right](x)\ dx.$$

Again by Lemma 1, this is

$$\int_{-\infty}^{\infty} f(x)\ \ \overline{\mathcal{F}[\mathcal{F}[g]]}(-x)\ dx.$$

This is, by the inversion formula,

$$2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}\ dx.$$

□

**Corollary 1.** *The operator $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ preserves the $L^2$-norm on $\mathcal{S}$.*

## 3 The Fourier transform on $L^2$

**Definition 1.** *Let $f \in L^2$. Let $f_k$, $k = 1, 2, \ldots$, be Schwartz functions with*

$$\underset{k\to\infty}{\text{l.i.m.}}\, f_k = f,$$

*where* l.i.m. *stands for convergence in $L^2$. The Fourier transform of $f$ is defined by*

$$\hat{f} = \mathcal{F}[f] = \underset{k\to\infty}{\text{l.i.m.}}\, \hat{f}_k.$$

One must explain why this definition makes sense:

**Theorem 2.** *Let $f \in L^2$. Let $f_k$, $k = 1, 2, \ldots$, be Schwartz functions with*

$$\underset{k\to\infty}{\text{l.i.m.}}\, f_k = f.$$

*(a) The sequence $\hat{f}_k$ in $\mathcal{S}$ converges in $L^2$.*

*(b) Let also $\tilde{f}_k$, $k = 1, 2, \ldots$, be Schwartz functions with*

$$\underset{k\to\infty}{\text{l.i.m.}}\, \tilde{f}_k = f.$$

*Then the $L^2$-limits of the $\hat{f}_k$ and the $\hat{\tilde{f}}_k$ are the same.*

*Proof.* This immediately follows from the fact that $\frac{1}{\sqrt{2\pi}}\mathcal{F}$ preserves the $L^2$-norm on $\mathcal{S}$. □

## 4 The Fourier transform on $L^2$ preserves $L^2$-products up to $2\pi$

**Theorem 3.** *If $f$ and $g$ are $L^2$-functions, then*

$$\int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}\, dt = 2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

*Proof.* Let $\{f_k\}_{k=1,2,\ldots}$ and $\{g_k\}_{k=1,2,\ldots}$ be sequences in $\mathcal{S}$ with

$$\underset{k\to\infty}{\text{l.i.m.}}\, f_k = f, \qquad \underset{k\to\infty}{\text{l.i.m.}}\, g_k = g.$$

Then, denoting the $L^2$-inner product by $\langle\cdot,\cdot\rangle$,

$$\int_{-\infty}^{\infty} \hat{f}(t)\overline{\hat{g}(t)}\, dt = \langle \hat{f}, \hat{g}\rangle = \lim_{k\to\infty} \langle \hat{f}_k, \hat{g}_k\rangle = 2\pi \lim_{k\to\infty} \langle f_k, g_k\rangle = 2\pi\langle f, g\rangle =$$

$$2\pi \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx.$$

□

**Corollary 2.** *If $f \in L^2$, then*

$$\|\hat{f}\|_2 = \sqrt{2\pi}\, \|f\|_2.$$

**Corollary 3.** *Let $f$ and $f_k$, $k = 1, 2, \ldots$, be $L^2$-functions. Assume that*

$$\operatorname*{l.i.m.}_{k\to\infty} f_k = f,$$

*where* l.i.m. *stands for the $L^2$-limit. Then*

$$\operatorname*{l.i.m.}_{k\to\infty} \hat{f}_k = \hat{f}.$$

## 5 On $L^1 \cap L^2$, the two definitions coincide

**Theorem 4.** *Let $f \in L^1 \cap L^2$. Then $\hat{f} = \mathcal{F}[f]$ in the sense of Definition 1 is the same as*

$$\int_{-\infty}^{\infty} f(x) e^{itx}\, dx.$$

*Proof.* There are functions $\varphi_k \in \mathcal{S}$, $k = 1, 2, \ldots$, which converge to $f$ in both $L^1$ and $L^2$. (Construct $\varphi_k$ by convolving $f \cdot 1_{[-k,k]}$ with $k\psi(kx)$, where $\psi \in C_c^\infty$ is non-negative with integral 1, for instance.) Then

$$\mathcal{F}[f] = \lim_{k\to\infty} \mathcal{F}[\varphi_k]$$

by Definition 1, where on the right-hand side, $\mathcal{F}$ is to be understood in the $L^1$-sense, so

$$\mathcal{F}[\varphi_k] = \int_{-\infty}^{\infty} \varphi_k(x) e^{ixt}\, dx.$$

But also

$$\lim_{k\to\infty} \mathcal{F}[\varphi_k] = \lim_{k\to\infty} \int_{-\infty}^{\infty} \varphi_k(x)\, e^{ixt}\, dx = \int_{-\infty}^{\infty} f(x)\, e^{ixt}\, dx$$

because the $\varphi_k$ converge to $f$ in $L^1$. □

Combining Corollary 3 with Theorem 4, we find the following corollary.

**Corollary 4.** *Let $f \in L^2$. Then*

$$\hat{f} = \operatorname*{l.i.m.}_{k\to\infty} \int_{-k}^{k} f(x)\, e^{ixt}\, dx.$$

## 6 The inversion theorem

**Theorem 5.** *Let $f \in L^2$. Then*

$$f = \underset{r\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \int_{-r}^{r} \hat{f}(t)\, e^{-itx}\, dt = \frac{1}{\sqrt{2\pi}} \hat{\hat{f}}(-x).$$

*Proof.* Let $f_k \in \mathcal{S}$ with $\underset{k\to\infty}{\text{l.i.m.}}\, f_k = f$. Then $\underset{k\to\infty}{\text{l.i.m.}}\, \hat{\hat{f}}_k(-x) = \hat{\hat{f}}(-x)$ by Corollary 3. Therefore

$$f = \text{l.i.m.}\, f_k(x) = \underset{k\to\infty}{\text{l.i.m.}} \frac{1}{\sqrt{2\pi}} \text{l.i.m.}\, \hat{\hat{f}}_k(-x) = \frac{1}{\sqrt{2\pi}}\, \hat{\hat{f}}(-x).$$

□

## 7 The Fourier transform as an automorphism of L$^2$

**Theorem 6.** *$\mathcal{F} : L^2 \to L^2$ is a bijection.*

*Proof.* Since $\|\hat{f}\|_2 = \sqrt{2\pi}\, \|f\|_2$, $\hat{f} = 0$ implies $f = 0$. This means that $\mathcal{F}$ is injective. To show that it is surjective, let $f \in L^2$. Then for almost all $x$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \hat{\hat{f}}(-x) = \frac{1}{\sqrt{2\pi}} \mathcal{F}[\hat{f}(-t)](x),$$

so

$$f = \mathcal{F}\left[\frac{\hat{f}(-t)}{\sqrt{2\pi}}\right].$$

This shows that $f$ is the Fourier transform of an $L^2$-function. □

## 8 Symmetry

**Theorem 7.** *For all $f, g \in L^2$,*

$$\int_{-\infty}^{\infty} f(r)\hat{g}(r)\, dr = \int_{-\infty}^{\infty} \hat{f}(s) g(s)\, ds. \tag{1}$$

Notice that $f \in L^2$ and $\hat{g} \in L^2$ imply $f\hat{g} \in L^1$ by the Cauchy-Schwarz inequality, and similarly $\hat{f}g \in L^1$. This does not mean that $\mathcal{F}$ is Hermitian — it is not. There are no complex conjugates in (1).

*Proof.* Let $\varphi_k$ and $\psi_k$, $k \in \mathbb{N}$, be Schwartz functions that converge in $L^2$ to $f$ and $g$, respectively. Then

$$\left| \int_{-\infty}^{\infty} f(r)\hat{g}(r)\, dr - \int_{-\infty}^{\infty} f_k(r)\hat{g}_k(r)\, dr \right| =$$

$$\left| \int_{-\infty}^{\infty} f(r)\hat{g}(r)\, dr - \int_{-\infty}^{\infty} f_k(r)\hat{g}(r)\, dr + \int_{-\infty}^{\infty} f_k(r)\hat{g}(r)\, dr - \int_{-\infty}^{\infty} f_k(r)\hat{g}_k(r)\, dr \right| \leq$$

$$\|f - f_k\|_2 \|\hat{g}\|_2 + \|f_k\|_2 \|\hat{g} - \hat{g}_k\|_2 \tag{2}$$

by the Cauchy-Schwarz inequality. The expression in (2) converges to $0$ as $k \to \infty$. Therefore

$$\int_{-\infty}^{\infty} f(r)\hat{g}(r)\, dr = \lim_{k\to\infty} \int_{-\infty}^{\infty} f_k(r)\hat{g}_k(r)\, dr = \lim_{k\to\infty} \int_{-\infty}^{\infty} \hat{f}_k(s) g_k(s)\, ds.$$

This limit equals

$$\int_{-\infty}^{\infty} \hat{f}(s) g(s)\, ds$$

for the same reason for which $\int_{-\infty}^{\infty} f(r)\hat{g}(r)\, dr = \lim_{k\to\infty} \int_{-\infty}^{\infty} f_k(r)\hat{g}_k(r)\, dr$. □