Fourier transforms, part 1:
Fourier transforms of integrable functions

Christoph Börgers
Department of Mathematics, Tufts University
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Intended readers: At roughly the level of a first-year graduate student of mathematics.

Specific prerequisites include Lebesgue integration and some properties of L^1; Fatou's lemma and the dominated convergence theorem; the notion of Banach space; the open mapping theorem (I state it here, but I don't prove it)

Feedback: If you find this useful, or if you have comments or suggestions, or if you just want to say hello, I would very much enjoy hearing from you: cborgers@tufts.edu.

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1 Definition of the Fourier transform

The symbol L^1 always denotes the space of absolutely integrable functions $\mathbb{R} \to \mathbb{C}$ here, and $\|\cdot\|_1$ denotes the L^1-norm.

Definition 1. *Let $f \in L^1$. The* Fourier transform *of f is the function*

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{itx}\, dx, \quad t \in \mathbb{R}.$$

We also write $\mathcal{F}[f]$ instead of $\hat{f}$.

Often one sees the factor $\frac{1}{\sqrt{2\pi}}$ or the factor $\frac{1}{2\pi}$ in front of the integral, and often one sees a minus sign in the exponent. The notational convention I use here is in agreement with the usual definition of *characteristic functions* of probability measures, which are Fourier transforms of probability measures. (Part 5 of this series on the Fourier transform will discuss characteristic functions in detail.)

I will note a straightforward but useful fact. Let $f \in L^1$. The Fourier transform of the function $f(-x)$, evaluated at t, is the same as the Fourier transform of the function $f(x)$, evaluated at $-t$:

$$\widehat{f(-x)}(t) = \widehat{f(x)}(-t) \quad \text{for all } t.$$

Using $\mathcal{F}$ instead of hats:

$$\mathcal{F}[f(-x)](t) = \mathcal{F}[f(x)](-t) \quad \text{for all } t.$$

Here is the calculation that verifies this:

$$\widehat{f(-x)}(t) = \int_{-\infty}^{\infty} f(-x)e^{ixt}\, dx = \int_{-\infty}^{\infty} f(u)e^{-iut} du = \int_{-\infty}^{\infty} f(x)e^{-ixt} dx = \widehat{f(x)}(-t).$$

2 All functions in $\mathcal{F}(\mathbf{L}^1)$ belong to $\mathbf{C}_0$

I will denote by C_0 the set of all functions $R \to \mathbb{C}$ with limit 0 at $\pm\infty$ (that's what the subscript 0 refers to).

Proposition 1. *If $f \in L^1$, then $\hat{f} \in C_0$.*

Proof. For $t \in \mathbb{R}$ and $h \in \mathbb{R}$,

$$|f(t+h) - f(t)| = \left| \int_{-\infty}^{\infty} f(x)e^{itx}\left(e^{ihx} - 1\right)\, dx \right| \leq \int_{-\infty}^{\infty} |f(x)|\left|e^{ihx} - 1\right|\, dx.$$

This tends to zero as $h \to 0$ by the dominated convergence theorem. Therefore $\hat{f}$ is continuous. It remains to prove that

$$\forall f \in L^1 \qquad \lim_{|t|\to\infty} \hat{f}(t) = 0. \tag{1}$$

This is also called the *Riemann-Lebesgue lemma*.

First we prove the weaker statement that

$$\forall f \in C_c^{\infty} \qquad \lim_{|t|\to\infty} \hat{f}(t) = 0, \tag{2}$$

where C_c^∞ denotes the space of C^∞-functions from $\mathbb{R}$ into $\mathbb{C}$ with compact support. This is true because for $f \in C_c^\infty$ and $t \neq 0$,

$$\hat{f}(t) = \int_{-\infty}^{\infty} f(x)e^{ixt}\,dx = -\int_{-\infty}^{\infty} f'(x)\frac{e^{ixt}}{it}\,dx = \frac{i}{t}\,\mathcal{F}[f'(x)](t),$$

so

$$\left|\hat{f}(t)\right| \leq \frac{1}{|t|}\left|\mathcal{F}[f'(x)](t)\right| \leq \frac{1}{|t|}\int_{-\infty}^{\infty}|f'(x)|\,dx.$$

Now we use that C_c^∞ is dense in L^1: For $f \in L^1$ and $\varepsilon > 0$, there is a function $\varphi \in C_c^\infty$ so that $\|f - \varphi\|_1 \leq \varepsilon$. Now

$$\|f-\varphi\|_1 \leq \varepsilon \quad \Rightarrow \quad \|\hat{f} - \hat{\varphi}\|_\infty \leq \varepsilon.$$

Since by (2), $\hat{\varphi}(t) \to 0$ as $|t| \to \infty$, we conclude

$$\limsup_{|t|\to\infty} |\hat{f}(t)| \leq \varepsilon,$$

and since this is true for all $\varepsilon > 0$, (1) follows. □

Some easy observations about C_0:

Proposition 2. *(a) Functions in C_0 are uniformly continuous. (b) When equipped with the infinity norm, C_0 is a Banach space.*

Proof. (a) Let $g \in C_0$. Let $\varepsilon > 0$. There exists a $T > 0$ so that

$$|g(t)| \leq \frac{\varepsilon}{2} \quad \text{if } |t| \geq T.$$

Since g is uniformly continuous on $[-2T, 2T]$, there exists a $\delta > 0$ so that for $t_1, t_2 \in [-2T, 2T]$ with $|t_1 - t_2| \leq \delta$, we have $|g(t_1) - g(t_2)| \leq \varepsilon$. Without loss of generality, $\delta \leq T$. It $t_1, t_2 \in \mathbb{R}$ and $|t_1 - t_2| \leq \delta$, then either both t_1 and t_2 lie in $[-2T, 2T]$, in which case $|g(t_1) - g(t_2)| \leq \varepsilon$, or both t_1 and t_2 lie outside $[-T, T]$, in which case $|g(t_1) - g(t_2)| \leq |g(t_1)| + |g(t_2)| \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. This proves (a). (b) is straightforward □

3 Not all functions in $\mathcal{F}(\mathbf{L}^1)$ belong to $\mathbf{L}^1$

A simple counterexample is $f(x) = 1_{[-1,1]}$. Its Fourier transform is, for $t \neq 0$,

$$\int_{-1}^{1} e^{itx}\,dx = \int_{-1}^{1}\cos(tx)dx = \left[\frac{\sin(tx)}{t}\right]_{x=-1}^{x=1} = 2\,\frac{\sin t}{t},$$

which isn't an L^1-function.

4 Symmetry of the Fourier transform

Theorem 1. *For $f \in L^1$ and $g \in L^1$,*

$$\int_{-\infty}^{\infty} f(r)\hat{g}(r)\,dr = \int_{-\infty}^{\infty} \hat{f}(s)g(s)\,ds. \tag{3}$$

The formula makes sense because $\hat{f}$ and $\hat{g}$ are bounded continuous functions, therefore $\hat{f}g$ and $f\hat{g}$ are L^1-functions. I use the variable names r and s instead of x or t here, because x is typically reserved for the independent variable of f and g, while t is reserved for the independent variable of $\hat{f}$ and $\hat{g}$, but here a single independent variable name is needed for f and $\hat{g}$, and similarly for $\hat{f}$ and g.

One can read the formula as saying that the Fourier transform is *symmetric* (but not *Hermitian*, since there are no complex conjugates in (3)).

Proof.

$$\int_{-\infty}^{\infty} \hat{f}(s)g(s)\,ds = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} f(r)e^{irs}\,dr \right) g(s)\,ds.$$

Since $f(r)g(s)e^{irs}$ is an absolutely integrable function of (r,s), we may re-write the nested integral as

$$\int_{-\infty}^{\infty} f(r) \left(\int_{-\infty}^{\infty} g(s)e^{irs}\,ds \right) dr = \int_{-\infty}^{\infty} f(r)\hat{g}(r)\,dr.$$

□

Symmetry turns out to be the key property needed for proving the Fourier inversion theorem, as I will now explain.

5 The Fourier inversion theorem for L^1-functions

Theorem 2. *If $f \in L^1$, then*

$$f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}\, e^{-\varepsilon^2 t^2}\,dt \quad \text{for almost all } x. \tag{4}$$

Before proving Theorem 2, I'll state an obvious consequence.

Corollary 1. *If $f \in L^1$ and $\hat{f} \in L^1$, then*

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t)e^{-itx}\,dt \quad \text{for almost all } x.$$

This follows from Theorem 2 by the dominated convergence theorem. We get f back from $\hat{f}$ by Fourier transforming $\hat{f}$ once more, but then putting a minus sign into the argument and dividing by 2π:

$$f(x) = \frac{\hat{\hat{f}}(-x)}{2\pi}.$$

This formula is valid only if both f and $\hat{f}$ are in L^1. However, (4) holds generally, under the assumption that $f \in L^1$.

Proof. For almost all $x \in \mathbb{R}$,

$$f(x) = \lim_{\varepsilon \to 0^+} \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} f(x+s) e^{-s^2/(2\sigma^2)}\, ds. \tag{5}$$

This is a well-known fact of analysis. Next we make the observation that

$$e^{-s^2/(2\sigma^2)} = \sigma\, \frac{\mathcal{F}[e^{-\sigma^2 t^2/2}](s)}{\sqrt{2\pi}}\ . \tag{6}$$

The proof is a simple calculation:

$$\mathcal{F}[e^{-\sigma^2 t^2/2}](s) = \int_{-\infty}^{\infty} e^{-\sigma^2 t^2/2} e^{its}\, dt = \int_{-\infty}^{\infty} e^{-\sigma^2 t^2/2 + its + s^2/(2\sigma^2)}\, dt\; e^{-s^2/(2\sigma^2)} =$$

$$\int_{-\infty}^{\infty} e^{-\left(\frac{\sigma t}{\sqrt{2}} + i\frac{s}{\sqrt{2}\,\sigma}\right)^2}\, dt\; e^{-s^2/(2\sigma^2)} = \ldots$$

(now substitute $u = \sigma t/\sqrt{2}$)

$$\ldots = \frac{\sqrt{2}}{\sigma} \int_{-\infty}^{\infty} e^{-\left(u + i\frac{s}{\sqrt{2}\,\sigma}\right)^2}\, du\; e^{-s^2/(2\sigma^2)} = \frac{\sqrt{2\pi}}{\sigma}\, e^{-s^2/(2\sigma^2)}.$$

This implies (6). Using (6) in (5), we find:

$$f(x) = \lim_{\sigma \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x+s)\; \mathcal{F}[e^{-\sigma^2 t^2/2}](s)\, ds \tag{7}$$

for almost all x. Now we use the symmetry property to move $\mathcal{F}$ from one factor in the Fourier transform to the other. So we have to take the Fourier transform of $f(x+s)$, seen as a function of s, and evaluate it at t. This yields

$$\int_{-\infty}^{\infty} f(x+s) e^{its}\, ds.$$

Substitute $u = x + s$:

$$\int_{-\infty}^{\infty} f(u) e^{it(u-x)}\, du = \hat{f}(t) e^{-itx}.$$

Therefore we obtain from (7):

$$f(x) = \lim_{\sigma \to 0^+} \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{f}(t) e^{-itx}\, e^{-\sigma^2 t^2/2}\, dt.$$

This is equivalent to (4). □

Corollary 2. *The linear map*

$$\mathcal{F}:\ L^1 \to C_0$$

is injective.

This follows immediately from the inversion theorem: When $\hat{f} = 0$, then $f = 0$.

6 Not all functions in $\mathbf{C}_0$ belong to $\mathcal{F}(\mathbf{L}^1)$

There is a very brief and abstract argument that implies that not all functions in C_0 belong to $\mathcal{F}(L^1)$, and in fact even a compactly supported continuous function need not belong to $\mathcal{F}(L^1)$. The argument does not yield a counterexample, it just proves that one must exist. It is based on the open mapping theorem from functional analysis, which I will state but not prove here.

Theorem 3 (Open Mapping Theorem)**.** *Let X and Y be Banach spaces, and let*

$$F:\ X \to Y$$

be a continuous linear mapping. If F is surjective, then F maps open sets in X onto open sets in Y. In particular, if F is bijective, its inverse is continuous.

We denote by $C_c([-1,1])$ the set of continuous functions $\mathbb{R} \to \mathbb{C}$ with support contained in $[-1,1]$.

Theorem 4. *There are functions in $C_c([-1,1])$ that are not Fourier transforms of L^1-functions.*

Proof. C_0, equipped with the L^∞-norm, is a Banach space, and $C_c([-1,1])$ is a closed subspace of it, so itself a Banach space. The mapping

$$\mathcal{F}:\ L^1 \to C_0$$

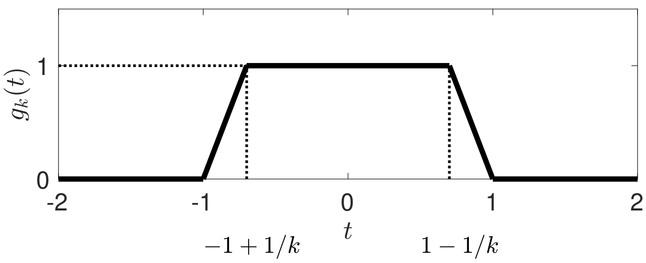
is continuous. Therefore the pre-image of $C_c([-1,1])$ is a closed subspace of L^1. We call this subspace V. Equipped with the L^1-norm, it is a Banach space. So

$$\mathcal{F}:\ V \to C_c([-1,1])$$

is a continuous linear map, and it is injective. We will prove that it cannot be surjective.

Suppose it were surjective. Then by the open mapping theorem, its inverse would be continuous. We will prove that this cannot be the case, by producing a sequence $\{g_k\}_{k=1,2,...}$ in $C_c([-1,1])$ with $g_k = \hat{f}_k$, $f_k \in L^1$, and $\|g_k\|_\infty = 1$ for all k, but $\{\|f_k\|_1\}_{k=1,2,...}$ is unbounded.

Here is the function g_k:



We have

$$\hat{g}_k(x) = \int_{-1}^{1} g_k(t)\ e^{ixt}\ dt = \int_{-1}^{1} g_k(t)\ \cos(xt)\ dt = 2\int_0^1 g_k(t)\ \cos(xt)\ dt =$$

$$2\int_0^{1-1/k} \cos(xt)\ dt + 2k\int_{1-1/k}^{1} (1-t)\ \cos(xt)\ dt = \frac{2k}{x^2}\left(\cos\left(\left(1-\frac{1}{k}\right)x\right) - \cos x\right).$$

This is an L^1-function. From the Fourier inversion theorem, we now conclude

$$g_k(t) = \frac{1}{2\pi}\hat{\hat{g}}_k(-t) = \frac{1}{2\pi}\mathcal{F}(\hat{g}_k(x))(-t) = \frac{1}{2\pi}\mathcal{F}(\hat{g}_k(-x))(t) =$$

$$\mathcal{F}\left(\frac{k}{\pi x^2}\left(\cos\left(\left(1-\frac{1}{k}\right)x\right) - \cos x\right)\right)(t)$$

So the g_k are the Fourier transforms of the L^1-functions

$$f_k(x) = \frac{k}{\pi x^2}\left(\cos\left(\left(1-\frac{1}{k}\right)x\right) - \cos x\right).$$

By Fatou's Lemma,

$$\liminf_{k\to\infty}\|f_k\|_1 = \liminf_{k\to\infty}\int_{-\infty}^{\infty}\frac{k}{\pi x^2}\left|\cos\left(\left(1-\frac{1}{k}\right)x\right) - \cos x\right|\ dx \geq$$

$$\int_{-\infty}^{\infty}\liminf_{k\to\infty}\frac{k}{\pi x^2}\left|\cos x - \cos\left(\left(1-\frac{1}{k}\right)x\right)\right|\ dx.$$

Now

$$\cos x - \cos\left(\left(1-\frac{1}{k}\right)x\right) = \sin\left(x - \frac{c}{k}x\right)\frac{x}{k}$$

for some $c \in (0,1)$. Therefore

$$\liminf_{k\to\infty}\frac{k}{\pi x^2}\left|\cos\left(\left(1-\frac{1}{k}\right)x\right) - \cos x\right| = \lim_{k\to\infty}\frac{k}{\pi x^2}\left|\cos\left(\left(1-\frac{1}{k}\right)x\right) - \cos x\right| = \frac{1}{\pi}\left|\frac{\sin x}{x}\right|.$$

So

$$\int_{-\infty}^{\infty}\liminf_{k\to\infty}\frac{k}{\pi x^2}\left|\cos\left(\left(1-\frac{1}{k}\right)x\right) - \cos x\right|\ dx = \infty$$

and therefore

$$\liminf_{k\to\infty}\|f_k\|_1 = \infty.$$

□

This is a non-constructive proof. I did not produce a counterexample, I just proved that one must exist. It's not a matter of behavior at infinity: There are continuous functions with compact support that do not belong to $\mathcal{F}(L^1)$.

7 Nice functions in C_0 belong to $\mathcal{F}(L^1)$

Definition 2. *We call a continuous function $g: \mathbb{R} \to \mathbb{C}$* piecewise C^2 *if either it is twice continuously differentiable, or there are numbers*

$$-\infty < t_1 < \ldots < t_N < \infty$$

so that g is C^2 on the intervals $I_0 = (-\infty, t_1]$, $I_1 = [t_1, t_2]$, $\ldots$, $I_{N-1} = [t_{N-1}, t_N]$, $I_N = [t_N, \infty)$.

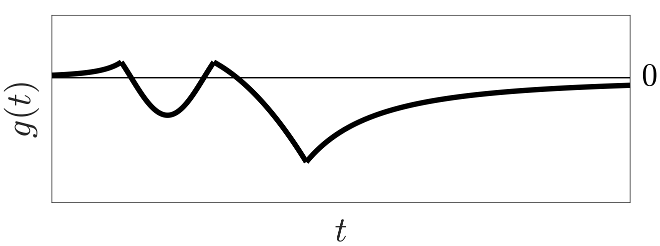
Proposition 3. *Let $g \in C_0$, and make the following additional assumptions.*

(i) $g \in L^1$,

(ii) $\lim_{|t| \to \infty} g'(t) = 0$,

(iii) *g is piecewise C^2, and on each of the subintervals on which g is C^2, g'' is absolutely integrable.*

Then g is the Fourier transform of an L_1-function.



Proof. It is enough to show that $\hat{g} \in L^1$. Namely, in that case

$$g(t) = \frac{1}{2\pi}\hat{\hat{g}}(-t) = \mathcal{F}\left(\frac{\hat{g}(x)}{2\pi}\right)(-t) = \mathcal{F}\left(\frac{\hat{g}(-x)}{2\pi}\right)(t),$$

so g is the Fourier transform of the L^1-function $\frac{\hat{g}(-x)}{2\pi}$.

Let $I_0, \ldots, I_N$ be the intervals as in Definition 2. If $g \in C^2$, let $N = 0$ and $I_0 = \mathbb{R}$. Then

$$\hat{g}(x) = \sum_{j=0}^{N} \int_{I_j} g(t) e^{itx}\, dt = -\sum_{j=0}^{N} \int_{I_j} g'(t) \frac{e^{itx}}{ix}\, dt.$$

(The boundary terms at $\pm\infty$ vanish because $g \in C_0$, and the interior boundary terms cancel.) Integrating by parts once more, we obtain boundary terms that no longer cancel (although the ones at $\pm\infty$ still vanish because of condition (b)), and integral terms. All terms are $O(1/x^2)$ by assumption (c), so $\hat{g} \in L^2$. □

8 The Wiener algebra

Definition 3. *The subspace*

$$A = \mathcal{F}(L^1) \subsetneq C_0$$

is called the Wiener algebra.

With respect to the L^∞-norm, A s dense in C_0, since $C_c^\infty \subseteq A$ and even C_c^∞ is dense in C_0. Therefore A isn't closed topologically; its closure is all of C_0. It is, however, closed with respect to multiplication; this is the content of the next theorem. (This property is what makes it an *algebra* — a vector space together with a multiplication that's compatible with the vector space operations in the obvious ways.)

Theorem 5. *If $g_1 \in A$ and $g_2 \in A$, then $g_1 \cdot g_2 \in A$.*

Proof. Suppose f_1 and f_2 are L^1-functions with $g_1 = \hat{f}_1$ and $g_2 = \hat{f}_2$. We will first prove that the convolution

$$f_1 * f_2(x) = \int_{-\infty}^{\infty} f_1(x-y) f_2(y)\, dy$$

is an L^1-function. In fact,

$$\int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} f_1(x-y) f_2(y)\, dy \right| dx \le \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f_1(x-y)|\, |f_2(y)|\, dy \right) dx.$$

By Tonelli's theorem, this equals

$$\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} |f_1(x-y)|\, dx \right) `|f_2(y)|\, dy = \|f_1\|_1\, \|f_2\|_1.$$

The Fourier transform of $f_1 * f_2$ is

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x-y) f_2(y)\, dy\, e^{ixt}\, dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_1(x-y) e^{i(x-y)t} f_2(y) e^{iyt}\, dy\, dx.$$

By Fubini's theorem, this equals

$$\int_{-\infty}^{\infty} f_1(x-y) e^{i(x-y)t}\, dx \int_{-\infty}^{\infty} f_2(y) e^{iyt}\, dy = \hat{f}_1(t)\, \hat{f}_2(t) = g_1(t)\, g_2(t).$$

□