Differentiation under the integral sign

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Intended readers: At roughly the level of a first-year graduate student of mathematics.

Specific prerequisites include sets of measure zero, L^1, the dominated convergence theorem

Feedback: If you find this useful, or if you have comments or suggestions, or if you just want to say hello, I would very much enjoy hearing from you: cborgers@tufts.edu.

Introduction

Suppose that $I \subseteq \mathbb{R}$ and $J \subseteq \mathbb{R}$ are intervals. Let $f = f(x,y)$ be a real function on $I \times J$. The statement

$$\forall x \in I \quad \frac{d}{dx}\int_J f(x,y)\,dy = \int_J \frac{\partial f}{\partial x}(x,y)\,dy \tag{1}$$

is a continuous version of the sum rule of calculus, so one expects it to hold.

A striking fact of analysis is that it does not always hold. You probably learned this in an analysis course. I know I did, but it never sank in completely. I am going to attempt to explain it here so thoroughly that you won't ever forget it again, or at least so thoroughly that I myself won't. Examples of functions f for which (1) fails to be true can feel a bit contrived. I want to see how you or I might have come up with those examples, and whether there are simple ones.

There are conditions that need to be satisfied for (1) to even *make sense*. Let me record these conditions first:

(a)

$$\forall x \in I \quad f(x,\cdot) \in L^1(J)$$

(b)

$$\forall x \in I \quad \exists \mathcal{N}_x \subseteq \mathbb{R} \text{ of measure zero} \quad \forall y \in J - \mathcal{N}_x \quad \frac{\partial f}{\partial x}(x,y) \quad \text{exists}$$

$$\text{and} \quad \frac{\partial f}{\partial x}(x,\cdot) \in L^1(J - \mathcal{N}_x)$$

I wrote $L^1(J-\mathcal{N}_x)$ in (b), which of course is the same as $L^1(J)$, since $\mathcal{N}_x$ is a set of measure zero, and L^1-"functions" are really equivalence classes of functions that differ only on sets of measure zero. I wrote $L^1(J-\mathcal{N}_x)$ anyway to emphasize that I don't want to

consider distributional derivatives here. For instance, the function $f(x)$ with $f(x) = 0$ for $x \le 0$ and $f(x) = 1$ for $x > 0$ is differentiable on $\mathbb{R} - \{0\}$, and its derivative on $\mathbb{R} - \{0\}$ is zero. In the context of this discussion, I would view its derivative as the L^1-function 0, even though of course distributionally, its derivative is the Dirac δ-distribution.

What ChatGPT told me

I asked ChatGPT to give me a simple example of intervals I and J, and a function $f = f(x,y)$ defined on $I \times J$, so that (a) and (b) hold, but (1) doesn't. ChatGPT answered: $I = J = \mathbb{R}$, and

$$f(x,y) = \begin{cases} e^{-y} & \text{if } y > x, \\ 0 & \text{if } y \le x. \end{cases}$$

Indeed, (a) is certainly correct, and so is (b), with $\mathcal{N}_x = \{x\}$. For $y \notin \mathcal{N}_x$, namely for $y \neq x$, we have $\frac{\partial f}{\partial x}(x,y) = 0$. Furthermore,

$$\int_{-\infty}^{\infty} f(x,y)\, dy = \int_{x}^{\infty} e^{-y}\, dy = e^{-x}.$$

Therefore

$$\frac{d}{dx}\int_{-\infty}^{\infty} f(x,y)\, dy = -e^{-x} \neq 0 = \int_{\mathbb{R}-\mathcal{N}_x} \frac{\partial f}{\partial x}(x,y)\, dy.$$

ChatGPT did what I asked it to do, but the example works only because we insist on not using distributional derivatives, considering $\frac{\partial f}{\partial x}(x,y)$ for $x \neq y$ only. In fact, one should really say that

$$\frac{\partial f}{\partial x}(x,y) = -e^{-y}\, \delta(x-y)$$

where $\delta(x-y)$ denotes the delta distribution at y, as a (generalized) function of x. If we interpreted $\frac{\partial f}{\partial x}(x,y)$ this way, then $\int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(x,y)\, dy$ would become $-e^{-x}$, so (1) would hold.

What "Counterexamples in Analysis" told me

Not quite satisfied with this example, I consulted *Counterexamples in Analysis* [1]. There I found the following example.

$$I = J = [0,\infty), \quad f(x,y) = \frac{x^3}{y^2} e^{-x^2/y},$$

understood to be 0 when $y = 0$. It is easy to verify that (a) holds. Second, (b) holds with $\mathcal{N}_x$ equal to the empty set:

$$\frac{\partial f}{\partial x}(x,y) = \left(3\frac{x^2}{y^2} - 2\frac{x^4}{y^3}\right) e^{-x^2/y},$$

understood to be 0 if $y = 0$. This function is integrable from $y = 0$ to $y = \infty$ for any fixed $x \ge 0$:

$$\int_0^{\infty} f(x,y)\, dy = \int_0^{\infty} \frac{x^3}{y^2} e^{-x^2/y}\, dy = \left[x e^{-x^2/y}\right]_{y=0}^{y=\infty} = x.$$

So

$$\frac{d}{dx}\int_0^\infty f(x,y)\,dy = 1 \neq 0 = \int_0^\infty \frac{\partial f}{\partial x}(0,y)\,dy.$$

This is a good example, but it leaves me without understanding. Why does it work? It seems like a randomly chosen function that happens to work. It is infinitely often differentiable on $[0,\infty)\times[0,\infty)-\{(0,0)\}$, but it is not continuous at $(0,0)$. For instance, as $(x,y)\to(0,0)$ with $y=x^2$, $f(x,y)=(ex)^{-1}\to\infty$.

I want to explain here how one could come up with this example, and then ask whether there are similar examples that are continuous everywhere, or even smooth everywhere.

A standard theorem

Theorem 1. *Let $I\subseteq\mathbb{R}$ and $J\subseteq\mathbb{R}$ be intervals, and let $f=f(x,y)$ be a measurable function so that*
(a)

$$\forall x\in I \quad f(x,\cdot)\in L^1(J)$$

and
(b')

$$\exists \mathcal{N}\subseteq\mathbb{R} \text{ of measure zero} \quad \forall x\in I \quad \forall y\in J-\mathcal{N} \quad \frac{\partial f}{\partial x}(x,y) \text{ exists}$$

and is a measurable function of $y\in J-\mathcal{N}$, and

$$\exists g\in L^1(J-\mathcal{N}) \quad \forall x\in I \quad \left|\frac{\partial f}{\partial x}(x,y)\right|\le g(y) \quad \text{for almost all } y\in J-\mathcal{N}. \qquad (2)$$

Then (1) *holds.*

Proof.

$$\frac{d}{dt}\int_I f(x,y)\,dy = \lim_{h\to 0}\frac{\int_J f(x+h,y)\,dy - \int_J f(x,y)\,dy}{h} = \lim_{h\to\infty}\int_J \frac{f(x+h,y)-f(x,y)}{h}\,dy.$$

For $y\in J-\mathcal{N}$, we have

$$\left|\frac{f(x+h,y)-f(x,y)}{h}\right| = \left|\frac{\partial f}{\partial x}(x+c(x,y,h)h,y)\right|\le g(y)$$

for some function $c(x,y,h)$ about which we know nothing other than that its values lie in $(0,1)$. The dominated convergence theorem implies the assertion. □

If we are just interested in showing that

$$\frac{d}{dx}\int_J f(x,y)\,dy\bigg|_{x=x_0} = \int_J \frac{\partial f}{\partial x}(x_0,y)\,dy$$

for some particular $x_0 \in I$, then of course we can replace $\forall x \in I$ in (2) by $\forall x \in I \cap [x_0 - \varepsilon, x_0 + \varepsilon]$ for any $\varepsilon > 0$, since we can replace I by the smaller interval $I \cap [x_0 - \varepsilon, x_0 + \varepsilon]$.

In summary, if the integrability of $\frac{\partial f}{\partial x}(x, \ldots)$ is *locally uniform in* x, then the dominated convergence theorem implies that differentiation under the integral sign is valid.

Counterexamples on the unit square

If you wanted to construct a function $f = f(x,y)$ for which (a) and (b) hold, but (b') does not, how would you go about it? Let's take $I = J = [0,1]$ first to keep things simple. We will try to construct f so that (1) fails at $x = 0$. Therefore (b') should in fact fail for any $I = [0, \varepsilon]$ with $\varepsilon \in (0,1]$. In other words, for no $\varepsilon \in (0,1]$ should

$$\max_{0 \le x \le \varepsilon} \left| \frac{\partial f}{\partial x}(x,y) \right|$$

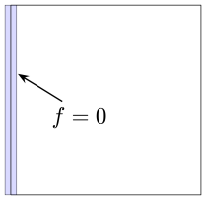
be an integrable function of y. This can be accomplished by making sure that

$$\forall \varepsilon \in (0,1] \quad \exists y_0 \in (0,1] \quad \forall y \in (0,y_0] \quad \max_{0 \le x \le \varepsilon} \left| \frac{\partial f}{\partial x}(x,y) \right| \ge \frac{1}{y}. \tag{3}$$

By the mean value theorem, (3) would follow from

$$\forall \varepsilon \in (0,1] \quad \exists y_0 \in (0,1] \quad \forall y \in (0,y_0] \quad \exists x \in (0,\varepsilon] \quad \left| \frac{f(x,y) - f(0,y)}{x} \right| \ge \frac{1}{y}. \tag{4}$$

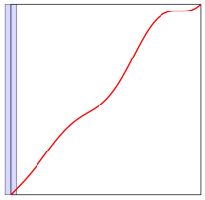
To keep things as simple as possible, we will define f in such a way that $f(0,y) = 0$ for $y \in [0,1]$.



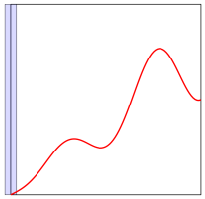
Then condition (4) becomes

$$\forall \varepsilon \in (0,1] \quad \exists y_0 \in (0,1] \quad \forall y \in (0,y_0] \quad \exists x \in (0,\varepsilon] \quad |f(x,y)| \ge \frac{x}{y}. \tag{5}$$

This suggests the following idea. Consider a continuous curve in the unit square that runs into the origin, like this for instance:



or even like this:



On this curve, define $f(x,y) = \frac{x}{y}$. Regardless of how we extend f to the rest of the unit square (remembering, of course, to satisfy the condition $f(0,y) = 0$ for all $y \in [0,1]$), (5) will hold, and therefore (b') will fail with $I = [0,\varepsilon]$ for any $\varepsilon \in (0,1]$. Of course, that doesn't yet mean that (1) fails; it just means that there is a chance that (1) fails. We will still have to verify that it does.

I will now define a particular family of curves, and a particular way of extending f from the curve to the unit square, and prove that for those choices, indeed (1) fails. Take the curve to be the graph of a function

$$\gamma\colon\ [0,1] \to [0,1],$$

that is infinitely often differentiable on $(0,1]$, continuous at 0 with $\gamma(0) = 0$, and $\gamma(x) > 0$ for $x > 0$.[1]

As already stated, we define $f(x,y) = \frac{x}{y}$ on the graph of γ. However, for a fixed $x > 0$, $\frac{x}{y}$ is not integrable, so (a) would be violated if we defined $f(x,y) = \frac{x}{y}$ everywhere. A simple way of addressing this is to multiply by $e^{1-\gamma(x)/y}$, which is 1 on the curve $y = \gamma(x)$. So our definition of f would now become

$$f(x,y) = \begin{cases} \frac{x}{y}\, e^{1-\gamma(x)/y} & \text{if } y > 0, \\ 0 & \text{if } y = 0. \end{cases}$$

This insures that (a) is satisfied. Is (b) satisfied? For $y = 0$, of course $\frac{\partial f}{\partial x}(x,y) = 0$. For $y > 0$ and $x > 0$,

$$\frac{\partial f}{\partial x}(x,y) = \left(\frac{1}{y} - \frac{x\gamma'(x)}{y^2}\right) e^{1-\gamma(x)/y}.$$

This is an integrable function of $y \in (0,1]$. Since we don't assume that $\gamma'(0)$ exists, this formula is not necessarily valid at $x = 0$. It is, however, easy to verify directly from the definition of the derivative that

$$\frac{\partial f}{\partial x}(0,y) = \frac{e}{y}$$

for $y > 0$. This is not an L^1-function, so (b) fails at $x = 0$. We must modify our definition of f one more time.

[1] I don't want to assume that γ is differentiable at 0 because functions such as $\gamma(x) = \sqrt{x}$ yield interesting examples, as discussed later.

We multiply by another factor equal to 1 on the curve $y = \gamma(x)$, chosen so that $\frac{\partial f}{\partial x}(0,y)$ simply becomes 0 for all y. The simplest way of doing this is to multiply by $\frac{\gamma(x)}{y}$. So now out definition becomes

$$f(x,y) = \begin{cases} \frac{x\,\gamma(x)}{y^2}\, e^{1-\gamma(x)/y} & \text{if } y > 0, \\ 0 & \text{if } y = 0. \end{cases} \tag{6}$$

It is straightforward to verify, using the definition of the derivative, that now

$$\frac{\partial f}{\partial x}(0,y) = 0$$

for all $y \in [0,1]$, using that $\gamma(0) = 0$ and the continuity of γ at 0. (Differentiability of γ at 0 is not needed.) It is easy to check that the extra factor of $\frac{\gamma(x)}{y}$ does not disturb the integrability of $\frac{\partial f}{\partial x}(x,\cdot)$ for $x > 0$.

So now the function f is infinitely often differentiable on $(0,1] \times [0,1]$, and continuous on $[0,1] \times [0,1] - \{0,0\}$. Whether it is continuous at $(0,0)$ depends on the choice of γ; I will discuss that shortly.

First, I'll make one last modification: Divide (6) by e, since the factor of e doesn't affect any of the properties we are interested in. So our definition becomes

$$f(x,y) = \frac{x\,\gamma(x)}{y^2} e^{-\gamma(x)/y},$$

understood to be 0 for $y = 0$.

For any choice of γ satisfying our assumptions, f satisfies (a) and (b), but not (by design) (b') for any $I = [0,\varepsilon]$ with $\varepsilon \in (0,1]$. Furthermore,

$$\int_0^1 f(x,y)\, dy = \int_0^1 \frac{x\,\gamma(x)}{y^2} e^{-\gamma(x)/y} = \left[x e^{-\gamma(x)/y} \right]_{y=0}^{y=1} = x e^{-\gamma(x)}.$$

At $x = 0$, this has derivative 1. (Again use the definition of the derivative rather than the product and chain rules to see this, since γ need not be differentiable at 0.) On the other hand,

$$\int_0^1 \frac{\partial f}{\partial x}(0,y)\, dy = 0.$$

So here we have a situation where differentiation under the integral sign is not valid. I will summarize this discussion in the following theorem.

Theorem 2. *Let* $\gamma\colon [0,1] \to [0,1]$ *be continuous with* $\gamma(0) = 0$*, and assume that* γ *is infinitely often differentiable on* $(0,1]$ *and* $\gamma(x) > 0$ *for* $x > 0$*. Then the function*

$$f(x,y) = \frac{x\gamma(x)}{y^2} e^{-\gamma(x)/y},$$

understood to be zero when $y = 0$*, is continuous on* $[0,1]^2 - \{(0,0)\}$*, infinitely often differentiable on* $(0,1] \times [0,1]$*, and satisfies*

$$\frac{d}{dx} \int_0^1 f(x,y)\, dy \bigg|_{x=0} = 1, \quad \textit{but} \quad \int_0^1 \frac{\partial f}{\partial x}(0,y)\, dy = 0.$$

The example from *Counterexamples in Analysis* [1] mentioned earlier is the special case $\gamma(x) = x^2$:

$$f(x,y) = \frac{x^3}{y^2} e^{-x^2/y}.$$

(The intervals I and J were taken to be $[0,\infty)$ in [1], but that's immaterial here.) This function is infinitely often differentiable on $[0,1]^2 - \{(0,0)\}$, but not continuous at $(0,0)$, and in fact not bounded as $(x,y) \to (0,0)$. To see this, think about the limit as $x \to 0$ with $y = x^2$.

A slightly simpler example is obtained with $\gamma(x) = x$:

$$f(x,y) = \frac{x^2}{y^2} e^{-x/y}.$$

This function, too, is infinitely often differentiable on $[0,1]^2 - \{(0,0)\}$, and discontinuous at $(0,0)$. (Think about the limits $x = 0$, $y \to 0$ and $x = y \to 0$.) It is bounded, since $q^2 e^{-q}$ is a bounded function of $q \in [0,\infty)$. Let's verify explicitly again that (1) is violated at $x = 0$:

$$\int_0^1 \frac{x^2}{y^2} e^{-x/y}\, dy = \left[x e^{-x/y} \right]_{y=0}^{y=1} = x e^{-x},$$

so

$$\frac{d}{dx} \int_0^1 \frac{x^2}{y^2} e^{-x/y}\, dy = (1-x) e^{-x}.$$

At $x = 0$, this is 1. On the other hand,

$$\frac{\partial}{\partial x} \left(\frac{x^2}{y^2} e^{-x/y} \right) = \left(\frac{2x}{y^2} - \frac{x^2}{y^3} \right) e^{-x/y}.$$

so

$$\int_0^1 \frac{\partial}{\partial x} \left(\frac{x^2}{y^2} e^{-x/y} \right) \bigg|_{x=0} dy = 0.$$

Finally think about the example $\gamma(x) = \sqrt{x}$:

$$f(x,y) = \frac{x\sqrt{x}}{y^2} e^{-\sqrt{x}/y}.$$

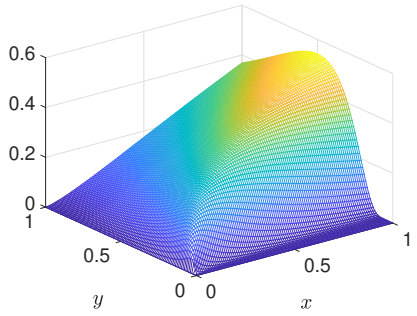
This function can be written as

$$f(x,y) = y\left(\frac{\sqrt{x}}{y}\right)^3 e^{-\sqrt{x}/y}.$$

Since $q^3 e^{-q}$ is a bounded function of $q \in [0,\infty)$, we find that

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0$$

in this example, so f is continuous even at $(0,0)$. Here is what it looks like:



The reasoning generalizes as follows.

Proposition 1. *Let $f = f(x,y)$ be defined as in Theorem 2.*

(a) If

$$\lim_{x\to 0} \frac{x}{\gamma(x)} = \infty,$$

then f is unbounded near $(0,0)$.

(b) If

$$\lim_{x\to 0} \frac{x}{\gamma(x)} = L$$

for some $L \in (0,\infty)$, then f is discontinuous at $(0,0)$, but bounded.

(c) If

$$\lim_{x\to 0} \frac{x}{\gamma(x)} = 0,$$

then f is continuous at $(0,0)$, and therefore continuous and bounded on all of $[0,1]^2$.

Proof. (a) If $\lim_{x\to 0} \frac{x}{\gamma(x)} = \infty$, then in the limit as $(x,y) \to (0,0)$ with $y = \gamma(x)$,

$$f(x,y) = \frac{x}{y} = \frac{x}{\gamma(x)}$$

tends to ∞.

(b) If $\lim_{x\to 0} \dfrac{x}{\gamma(x)} = L \in (0,\infty)$, then in the limit as $(x,y) \to (0,0)$ with $y = \gamma(x)$,

$$f(x,y) = \frac{x}{y} = \frac{x}{\gamma(x)}$$

tends to L. This implies that f is discontinuous at $(0,0)$, since $f(0,0) = 0$. For small enough $x > 0$, we have $x/\gamma(x) \le 2L$ and therefore

$$f(x,y) = \frac{x\gamma(x)}{y^2}\, e^{-\gamma(x)/y} = \frac{x}{\gamma(x)} \left(\frac{\gamma(x)}{y}\right)^2 e^{-\gamma(x)/y} \le 2L \max_{q\in[0,\infty)} q^2 e^{-q} = \frac{8L}{e^2} < \infty.$$

(c) If $\lim_{x\to 0} \dfrac{x}{\gamma(x)} = 0$, then

$$f(x,y) = \frac{x\gamma(x)}{y^2}\, e^{-\gamma(x)/y} = \frac{x}{\gamma(x)} \left(\frac{\gamma(x)}{y}\right)^2 e^{-\gamma(x)/y} \le \max_{q\in[0,\infty)} q^2 e^{-q}\, \frac{x}{\gamma(x)} = \frac{4}{e^2}\, \frac{x}{\gamma(x)}$$

tends to 0 as $(x,y) \to (0,0)$. □

Counterexamples on the entire positive quadrant

The examples of the previous section all work on the entire positive quadrant without modification:

Theorem 2'. *Let $\gamma\colon [0,\infty) \to [0,\infty)$ be continuous with $\gamma(0) = 0$, and assume that γ is infinitely often differentiable on $(0,\infty)$ and $\gamma(x) > 0$ for $x > 0$. Then the function*

$$f(x,y) = \frac{x\gamma(x)}{y^2} e^{-\gamma(x)/y},$$

understood to be zero when $y = 0$, is continuous on $[0,\infty)^2 - \{(0,0)\}$, infinitely often differentiable on $(0,\infty)\times[0,\infty)$, and satisfies

$$\frac{d}{dx}\int_0^\infty f(x,y)\,dy\bigg|_{x=0} = 1, \quad \textit{but} \quad \int_0^\infty \frac{\partial f}{\partial x}(0,y)\,dy = 0.$$

Proof.

$$\int_0^\infty f(x,y)\,dy = \int_0^\infty \frac{x\gamma(x)}{y^2}\, e^{-\gamma(x)/y}\,dy = \left[x e^{-\gamma(x)/y}\right]_{y=0}^{y=\infty} = x,$$

so

$$\frac{d}{dx}\int_0^\infty f(x,y)\,dy\bigg|_{x=0} = 1.$$

Since $\frac{\partial f}{\partial x}(0,y) = 0$ for all y,

$$\int_0^\infty \frac{\partial f}{\partial x}(0,y)\, dy = 0.$$

□

Counterexamples defined in the entire plane

We extend the examples of Theorems 2' to the entire plane, making the extension odd in x and even in y:

$$f(x,y) = \operatorname{sign}(x) \frac{|x|\gamma(|x|)}{|y|^2} e^{-\gamma(|x|)/|y|} = \frac{x\,\gamma(|x|)}{y^2} e^{-\gamma(|x|)/|y|}.$$

Because the extension is even in y, we have for $x \geq 0$

$$\int_{-\infty}^{\infty} f(x,y)\ dy = 2\int_0^\infty f(x,y)\ dy,$$

and because the extension is odd in x, we have for $x > 0$

$$\int_{-\infty}^{\infty} f(-x,y)\ dy = -\int_{-\infty}^{\infty} f(x,y)\ dy.$$

Therefore

$$\frac{d}{dx}\int_{-\infty}^{\infty} f(x,y)\ dy\Big|_{x=0} = 2$$

now.

Theorem 3. *Let $\gamma\colon [0,\infty) \to [0,\infty)$ be continuous with $\gamma(0) = 0$, and assume that γ is infinitely often differentiable on $(0,\infty)$ and $\gamma(x) > 0$ for $x > 0$. Then the function*

$$f(x,y) = \frac{x\gamma(|x|)}{y^2} e^{-\gamma(|x|)/|y|}, \qquad (x,y) \in \mathbb{R}^2,$$

understood to be zero when $y = 0$, is continuous on $\mathbb{R}^2 - \{(0,0)\}$, infinitely often differentiable on $\mathbb{R}^2 - \{(0,y) : y \in \mathbb{R}\}$, and satisfies

$$\frac{d}{dx}\int_{-\infty}^{\infty} f(x,y)\,dy\bigg|_{x=0} = 2, \quad \textit{but} \quad \int_{-\infty}^{\infty} \frac{\partial f}{\partial x}(0,y)\,dy = 0.$$

A smooth counterexample on the positive quadrant

It is clear that conditions (a) and (b') hold if J is a compact interval and $f \in C^1$. But are there smooth functions on $\mathbb{R}^2$ for which differentiation under the integral sign fails? The answer is yes, and one can find examples using ideas similar to the ones we have explored here. '

First, a smooth counterexample for $I = [0,\infty)$, $J = [0,\infty)$. For instance, suppose that $f(x,y) = \frac{x}{y}$ for all points (x,y) on the curve $y = 1/x$, but $f(0,y) = 0$. By the mean value theorem, $\frac{\partial f}{\partial x}(x,y) = 1/y$ for some x between 0 and $1/y$. This implies that condition (b') is violated at $x = 0$, for any $I = [0,\varepsilon]$ with $\varepsilon > 0$.

Of course, $\int_0^\infty \frac{x}{y}\,dy$ does not exist for a fixed positive x. We would also like the x-derivative to be 0 on the positive y-axis. A natural idea is to multiply by a function of the form

$$cxe^{-rx}$$

with $c > 0$ and $r > 0$ chosen so that the maximum of cxe^{-rx} occurs at $x = 1/y$, and the maximum value is 1: $r = y$ and $c = ey$. So we multiply by $exye^{-xy}$, obtaining

$$exye^{-xy}\frac{x}{y} = ex^2e^{-xy}.$$

The factor of e cannot matter, so we drop it.

Proposition 2. *Let* $f(x,y) = x^2 e^{-xy}$ *for* $x \geq 0$ *and* $y \geq 0$. *Then*

$$\int_0^\infty f(x,y)\,dy = x \quad \text{for all } x \geq 0$$

but

$$\int_0^\infty \frac{\partial f}{\partial x}(0,y) = 0.$$

A smooth counterexample defined in the entire plane

The following variation on the preceding example works on $\mathbb{R}^2$:

$$f(x,y) = x^2 e^{-(xy)^2}.$$

We have

$$\int_{-\infty}^\infty x^2 e^{-(xy)^2}\,dy = \sqrt{\pi}\,|x|.$$

Therefore

Proposition 3.

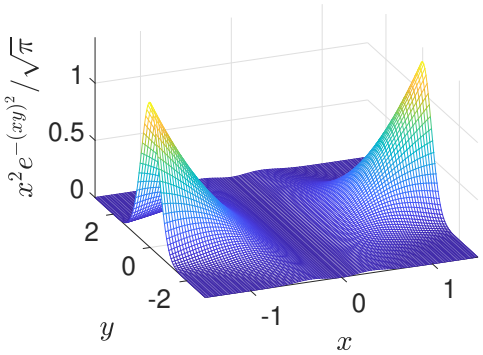
$$\int_{-\infty}^\infty \frac{\partial}{\partial x}\left(\frac{x^2 e^{-(xy)^2}}{\sqrt{\pi}}\right) dy = 0 \quad \text{at } x = 0,$$

but

$$\frac{d}{dx}\int_{-\infty}^\infty \frac{x^2 e^{-(xy)^2}}{\sqrt{\pi}}\,dy \quad \text{does not exist at } x = 0$$

since

$$\int_{-\infty}^\infty \frac{x^2 e^{-(xy)^2}}{\sqrt{\pi}}\,dy = |x|.$$



Why is condition (b') violated in this example? We have

$$\frac{\partial f}{\partial x}(x,y) = (2x - 2x^3y^2)e^{-(xy)^2}.$$

For $xy = 1/2$, this is

$$\frac{1}{y}\left(1 - \frac{1}{4}\right)e^{-1/4}.$$

This isn't an integrable function, so

$$\max_{x\in[0,\varepsilon]}\left|\frac{\partial f}{\partial x}(x,y)\right|$$

is not an integrable function of y for any $\varepsilon > 0$.

References

[1] B. GELBAUM AND J. OLMSTED, *Counterexamples in Analysis*, Dover Publicationa, 1992.