

Simple methods for single winner elections

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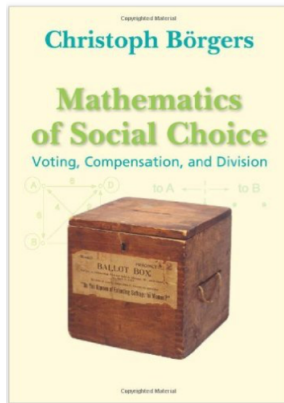
I have posted these slides there.

Why simple is good

To be politically useful, a method of voting must be simple and transparent.

We'll only think about single winner elections here: electing a president, governor, mayor, etc.

This talk is a sketch of the first half of my book on social choice theory, with some additions.



How it's usually done: plurality voting

- ▶ Every voter names their favorite candidate.
- ▶ Whoever is named most often wins.

Is this the best thing to do?

How does math get into this?

A single-winner election method is a map from the voters' preferences onto a winner.

Plurality voting isn't the only such map!

We can formulate desirable properties, and see which maps (single-winner election methods) have them.

What kind of properties would be desirable?

Perhaps the simplest example: The “one person, one vote” principle, a symmetry property:

If you and I swap votes, the outcome should not change.

Methods based on ranking, not grading

Each voter gives a strict ranking of all candidates.
We want to map the rankings onto a single winner.

Plurality voting falls into this framework. (We ignore everything about a voter's ranking except their top choice, and therefore we don't even collect anything but the top choice.)

Other voting methods use grading instead of ranking.
I will not discuss those.

Desirable properties

There are many. We'll focus on just four. All four express the idea that the majority should rule.

Stated softly:

1. If X is clearly the most popular candidate, X wins.
2. If X is clearly the least popular candidate, X loses.
3. If X is clearly less popular than Y , X loses.
4. No “spoilers”: Weak candidates remain inconsequential.

“Clearly most popular”

1. If X is clearly the most popular candidate, X wins.

Precisely:

1. If X would beat each other candidate in a two-person runoff,^{*}
 X wins.

Condorcet candidate, Condorcet criterion

^{*}We can tell once we know all the rankings!

Marquis de Condorcet, 1743–1794.



“Clearly least popular”

2. If X is clearly the least popular candidate, X loses.

Precisely:

2. If X would lose against each other candidate in a two-person runoff, X loses.

anti-Condorcet candidate, anti-Condorcet criterion

“Clearly less popular”

3. If X is clearly less popular than Y , X loses.

comparison criterion

Example: Assume 12 voters and 3 candidates, and X gets 5 first-place, 4 second-place, and 3 third-place votes. Y gets 5 first-place, 5 second-place, and 2 third-place votes, Then X should lose.

Note: Most pairs of candidates X , Y are not “comparable” in this sense. The comparison criterion only says something about the ones that are.

“Spoilers”

4. No “spoilers”: Weak candidates remain inconsequential.

We will make this one precise later.

How does plurality do on our four-question test?

1. If X is clearly the most popular candidate, X wins. **X**
(Condorcet criterion)
2. If X is clearly the least popular candidate, X loses. **X**
(anti-Condorcet criterion)
3. If X is clearly less popular than Y , X loses. **X**
(comparison criterion)
4. No “spoilers”: Weak candidates remain inconsequential. **X**

0 points out of 4

Which of my marks can you understand immediately?

Famous real-life example of a “spoiler”*:
Florida Presidential election 2000.

- ▶ George W. Bush (Republican): 48.847%
- ▶ Al Gore (Democrat): 48.838%
- ▶ Ralph Nader (Green): 1.635%



*I understand that one can reasonably argue otherwise.

Plurality violates both Condorcet criteria

Example: 12 people want to decide whether to eat Chinese, Japanese, or Indian food. Their preferences are as follows:

5	4	3
C	J	I
I	I	J
J	C	C

The Condorcet candidate is *I*.

By plurality, *C* wins. But *C* is the anti-Condorcet candidate!

Three simple alternatives to plurality voting

- ▶ instant runoff
(various U.S. politicians and organizations like it)
- ▶ Borda count
(hardly anybody seems to like it)
- ▶ majority rule
(Eric Maskin and Amartya Sen like it, and I think they are right)

Instant runoff

- ▶ Each voter ranks all candidates.
- ▶ Remove the one who is in first place *least* often.
- ▶ Consolidate: move lower-ranked candidates up a notch when the eliminated candidate has left a gap above them.
- ▶ Repeat until only one candidate is left standing.

5	4	3
C	J	I
I	I	J
J	C	C

Who wins here?

Political support for instant runoff

- ▶ Green Party
- ▶ Libertarian Party
- ▶ fairvote.org
- ▶ Maine ballot question November 2016
(passed 52% to 48%)
- ▶ San Francisco
(citywide offices, since 2004)
- ▶ Minneapolis
(city elections, since 2006)
- ▶ Saint Paul
(city elections, since 2009)
- ▶ Republican Party of Utah
(used instant runoff 2002–2004)
- ▶ ...

Borda count

number of candidates = n

- ▶ Each voter ranks all candidates.
- ▶ A candidate gets n points for a first-place ranking, $n - 1$ points for a second-place ranking, and so on.
- ▶ Whoever gets most points wins.

5	4	3
C	J	I
I	I	J
J	C	C

$$C: 5 \times 3 + 4 \times 1 + 3 \times 1 = 22$$

$$I: 5 \times 2 + 4 \times 2 + 3 \times 3 = 27$$

$$J: 5 \times 1 + 4 \times 3 + 3 \times 2 = 23$$

So *I* wins here.

Chevalier de Borda, 1733–1799



Actual uses of Borda count

Few:

- ▶ certain votes in the University of Michigan mathematics department
- ▶ Slovenia
- ▶ most valuable player in major league baseball
- ▶ ...

The majority rule framework

- ▶ Each voter ranks all candidates.
- ▶ If there is a Condorcet candidate, that candidate is declared the winner of the election.

The New York Times

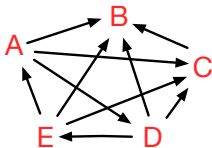
Sunday Review | OPINION

How Majority Rule Might Have Stopped Donald Trump

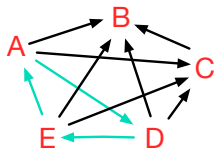
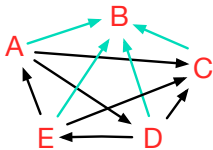
By ERIC MASKIN and AMARTYA SEN APRIL 28, 2016

We will think about majority rule more carefully now. Eventually we will end up with a specific majority rule method.

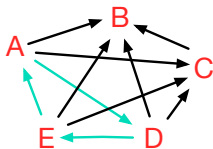
The pairwise comparison graph



There is no Condorcet candidate here, but there are an anti-Condorcet candidate and a Condorcet cycle:



Does the graph I drew actually represent
a possible election outcome?



THEOREM. *Every pairwise comparison graph represents a possible election outcome.*

PROOF.

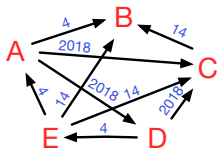
1. There are election outcomes in which all candidates tie.
2. For any X and Y , there are election outcomes in which all candidates tie, except X beats Y by a margin of 2.
3. Putting many such outcomes together, you can get any pairwise comparison graph. \square

In fact, the proof shows more:

THEOREM. Given any pairwise comparison graph and margins of victory of equal parity (all even or all odd), there is an election outcome generating the graph and margins.

If w_i are the weights (i labels edges), the minimum number of voters needed to realize the graph as an election outcome lies between $\max_i w_i$ and $\sum_i w_i$.

PROOF. A puzzle for your lunch break. \square



This can be realized as an election outcome with $3 \times 4 + 3 \times 14 + 3 \times 2018 = 6108$ voters, and it can certainly *not* be realized with fewer than 2018 voters.

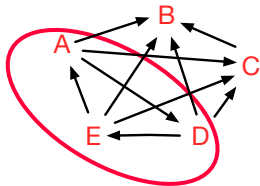
Smith candidates and non-Smith candidates

THEOREM.

1. For any pairwise comparison graph, there is a unique smallest non-empty set S such that every $X \in S$ beats every $Y \notin S$ in pairwise comparison.
2. If X is a Condorcet candidate, then $S = \{X\}$.

(John H. Smith, Boston College)

DEFINITION. We call the members of S the Smith candidates, and the others the non-Smith candidates.



Generalized majority rule, according to Smith

- ▶ Each voter ranks all candidates.
- ▶ A winner is picked from S .

We must of course still say how we pick the winner.

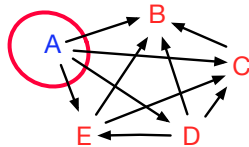
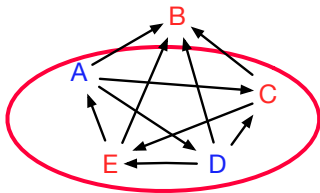
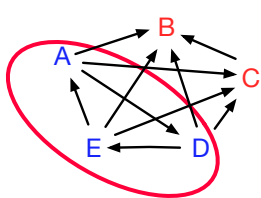
The method of pairwise comparison

- ▶ Each voter ranks all candidates.
- ▶ Compute the pairwise comparison graph.
- ▶ The candidate(s) with the most outgoing arrows win(s).

Not a good method: Ties are too likely, even for many voters.

However, this method is a useful stepping stone towards a good method, as we will now show.

THEOREM. Removal of any or all non-Smith candidates from all ballots affects neither S , nor the set of pairwise comparison winners. In particular, pairwise comparison winners are Smith candidates.



circled in red: Smith candidates
 blue: pairwise comparison winners

Finally, a precise definition of “spoiler”

DEFINITION. A spoiler is a non-Smith candidate who affects the election outcome.

A “spoiler-free” method is one with a description that starts, or could equivalently start, with “Remove all non-Smith candidates from all ballots.”

The method of pairwise comparison is spoiler-free.

My favorite majority rule method

- ▶ Each voter ranks all candidates.
- ▶ Compute the pairwise comparison winners, and remove all others from the ballots.
- ▶ For the reduced ballots, use Borda count.

In the second step, we could use the Smith candidates instead of the pairwise comparison winners, but pairwise comparison winners are easier to explain and compute than Smith candidates.

(“Easier to explain” is politically important.)

16 theorems in one table

	C.	anti-C.	comparison	spoiler
plurality	X	X	X	X
instant runoff	X	✓	X	X
Borda	X	✓	✓	X
majority*	✓	✓	X	✓

* any method that first removes all non-Smith candidates, then picks a single winner from S .

A second puzzle to solve over lunch: Prove that Borda count satisfies the anti-Condorcet criterion.

Problems with a real life instant runoff

2009 mayoral election in Burlington, VT.

The three main candidates were Bob Kiss (Progressive), Kurt Wright (Republican), Andy Montroll (Democrat).

The preferences were approximately like this:

1560	994	2327	655	1140	2158
M	M	K	K	W	W
K	W	M	W	K	M
W	K	W	M	M	K

Montroll was the Condorcet candidate, but Kiss won. Most of Wright's supporters preferred Montroll to Kiss, but managed to get their least favorite candidate elected by voting honestly.

FairVote declared this a "great success".

However, a year later, Burlington abolished instant runoff voting.

Spoilers in instant runoff

1. In contrast with plurality voting, very weak candidates never cause problems in instant runoff.
2. As soon as there are three or more candidates with roughly equally many first-place votes, spoiler problems re-appear.

There is no perfect voting method

There are many theorems to this effect! [Arrow's Theorem](#) was the first. The second-most famous is this one:

GIBBARD-SATTERTHWAITE THEOREM. No reasonable voting method can prevent successful dishonest (manipulative) voting.

Here is another one:

THEOREM. No election method satisfies the comparison criterion and the Condorcet criterion at the same time.

PROOF. Find an example of a Condorcet candidate who has to lose by the comparison criterion. This is your third and final lunch puzzle. \square

Where things get murky and therefore research should be done

- ▶ How easy is successful dishonest (manipulative) voting?
- ▶ How likely are violations of our criteria to occur in practice?

To my knowledge, the **probabilistic aspect of this subject** is not satisfactorily understood even for the simplest methods.

One has to think about random election outcomes, but with which distribution? (Note that the relevant distribution will shift when the election method shifts.)

Only the simplest methods are interesting if one wants any of this to be politically relevant. (And why else would one study it?)

Summary

A simple majority rule (Condorcet-fair) method may be best:

- ▶ Each voter ranks the candidates.
- ▶ Remove pairwise comparison losers.
- ▶ Borda count among the remaining candidatea.

I hesitate to say this with too much conviction without understanding the probabilistic aspect of this field yet.

Review of the three puzzles for lunch

1. Show that Borda count satisfies the anti-Condorcet criterion.
2. 2.1 Show that every pairwise comparison diagram in which all margins of victory have the same parity (all even, or all odd) can be realized as an election outcome.
2.2 Show that the minimum number of voters needed lies between the largest margin of victory and the sum of all the margins of victory.
2.3 Show by example that the minimum number of voters needed can be equal to the largest margin of victory, and can be equal to the sum of all the margins.
3. Find an example of a Condorcet candidate who loses by the comparison criterion.

E-mail your solutions to cborgers@tufts.edu.

(Just for fun, there will be no prizes for this...)

Thank you for listening!